THE QUOTIENT OF EXPONENTIAL POLYNOMIALS

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The purpose of this note is to furnish a new proof of a special case of a theorem of Ritt [3] on algebraic functions of exponential polynomials. In the proof we shall make use of some well-known properties of entire functions of order one and normal type.

For another proof on the theorem of Ritt see a paper by H. Selberg [4].

Definition. An exponential polynomial is a finite sum of the form

$$\sum_{j=1}^{m} a_j e^{\mu_j z}, \qquad a_j \neq 0, \qquad \mu_k \neq \mu_j \text{ for } k \neq j$$

and $\operatorname{Re} \mu_1 \geq \operatorname{Re} \mu_2 \geq \cdots \geq \operatorname{Re} \mu_n$.

Theorem. *If $A(z)$ and $B(z)$ are two exponential polynomials*

$$A(z) = \sum_{j=1}^{m} a_j e^{\mu_j z}, \qquad B(z) = \sum_{j=1}^{n} b_j e^{\nu_j z},$$

and if $f(z) = A(z)/B(z)$ is an entire function then $f(z)$ too is an exponential polynomial.

$A(z)$ and $B(z)$ are obviously of order one and of normal type. We begin our proof by showing that under the hypotheses of the theorem $f(z)$ too is of order one and normal type.

That $f(z)$ is of order one follows immediately from the canonical product representation for $A(z)$ and $B(z)$. To show that it is of normal type consider any direction φ such that the maxima

$$\max_{1 \leq j \leq m} \operatorname{Re}(\mu_j e^{i\varphi}), \qquad \max_{1 \leq j \leq n} \operatorname{Re}(\nu_j e^{i\varphi}) \tag{1}$$

are assumed for one value of j only. For such directions the sums $A(z)$ and $B(z)$ are dominated by their respective maximal terms for large $|z|$; therefore their ratio is certainly of exponential type along such half rays. By the Phragmén-Lindelöf principle, a function of order one which is of exponential type along two half rays enclosing an angle $\alpha < \pi$ is of exponential type inside the angle too. Since there are only a finite number of directions for which one of the maxima (1) is assumed for more than one value of j, we can easily find three directions φ_1, φ_2, φ_3 such that $\varphi_2 - \varphi_1 \equiv \varphi_3 - \varphi_2 \equiv \varphi_1 - \varphi_3 \equiv 2\pi/3 \pmod{2\pi}$ such that $f(z)$ is of exponential type along these half rays and consequently everywhere.

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