METRIC METHODS IN LINEAR INEQUALITIES

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1. **Introduction.** By "metric methods" is meant the association of distances with pairs of elements wherever such an association is possible and useful, and the resulting subordination of various fields of research to common ideas. The aim of this paper is to present evidence for the view that the theory of systems of linear, algebraic inequalities, in finitely many indeterminates, might be advantageously developed as part of distance geometry.

Geometric, if not metric, considerations have long been used as aids in dealing with certain aspects of the theory (for example, the existence of solutions), but so far as the writer is aware these devices have served merely as occasional tools and not as a means of developing in a unified manner the whole theory. An excellent bibliography is contained in the comprehensive account of the subject given in [4]. Some later references to the literature are given in [3].

The basis for our introduction of metric methods is furnished by associating with each system of inequalities two subsets C and $\Sigma(C)$ of the unit n-sphere $S_{n,1}$ obtained by metrizing convexly (that is, with shorter arc metric) the surface of the sphere of radius 1 in euclidean space of $n + 1$ dimensions. The theory of the system is then developed in terms of the mutual relations of those sets, the "coefficient" and "solution" sets, respectively, and certain other sets obtained from them. The system of inequalities itself, its algebraic nature, *etc.* have fulfilled their functions as soon as the sets C and $\Sigma(C)$ have been defined and the metric considerations become operative.

Several advantages are attached to such a procedure. In the first place, the geometrization of the theory serves to make intuitive some of the classical results and to suggest new ones. It leads to the discovery and exploitation of a symbolism (an algebra for the solution operator Σ) which materially assists in exposing the theory.

Instead of being mostly concerned with the solution aspect (that is, the determination of properties of C which insure that $\Sigma(C)$ is not null), it is seen that this problem is but one, and perhaps not the most interesting, of many that arise regarding properties of C that are necessary and sufficient to give $\Sigma(C)$ a desired character. Thus, for example, it might be asked what conditions on C insure that $\Sigma(C) = C$, or that $\Sigma^2(C) = C$, or that $\Sigma^3(C) = \Sigma(C)$.

Due to recent results in the metric geometry of the convex n-sphere and half n-sphere, the metric approach leads easily to the derivation of existence theorems for solutions of systems of inequalities. It happens that certain theorems concerning the covering of subsets of $S_{n,1}$ by hemispheres and small caps are

Received June 5, 1948; in revised form, August 2, 1948. Presented to the American Mathematical Society, December 30, 1947.