AN UPPER BOUND FOR THE GONTCHAROFF CONSTANT

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The Gontcharoff constant G is defined as the least upper bound of numbers c such that $f(z)$, analytic in $|z| < 1$, is necessarily identically zero if $f^{(n)}(a_n) = 0$, $n = 0, 1, 2, \cdots$, and $\limsup_{n\to\infty} n|a_n| < c$. The definition is similar to that of the Whittaker constant W, defined as the least upper bound of numbers c such that $f(z)$, an entire function of order 1 and type not exceeding 1, is necessarily identically zero if $f^{(n)}(a_n) = 0$, $n = 0, 1, 2, \cdots$, and $|a_n| \leq c$. Concerning W it is known that $.7199 < W < .7378$; the lower bound was obtained by N. Levinson [3] and the upper bound by S. S. Macintyre [4]. It was shown by Kakeya [2] that $G \geq \log 2 = .693\cdots$; and Gontcharoff [1] observed that $f(z) = 1/(1 + z^2)$ provides the inequality $G \leq \pi/2$; Gontcharoff conjectured that $G = \pi/2$. However, the example $f(z) = (1 - z)/(1 + z^2)$ shows that $G \leq \pi/4 = .785\cdots$, and the analogy between the two constants, together with the older and more easily established limitation $\log 2 \leq W \leq \pi/4$, leads one to suspect a close connection between the two constants.

Macintyre's upper bound was obtained by finding that function $f(z)$ satisfying $f'(z) = f(\omega z)$, $|\omega| = 1$, which has a zero nearest to the origin; the absolute value c of this zero is an upper bound for W. Here we shall show that the same number c is also an upper bound for G, so that we have $.693 < G < .7378$. This fact provides, of course, no information as to whether $G = W$, although it is an attractive conjecture that this is so. Presumably a computation using Levinson's method would lead to a considerable improvement of the lower bound for G.

Let $f(z)$ be Macintyre's function, of order 1 and type 1, such that $f'(z) = f(\omega z)$, $|\omega| = 1$, $f(z_0) = 0$, $|z_0| = c$. Let $L(z)$ be the Laplace transform of $f(z)$ and $F(z) = z^{-1}L(z^{-1})$. Then $F(z)$ is analytic in $|z| < 1$. We have

$$F(z) = \int_0^\infty e^{-u} f(uz)\, du.$$

Hence

$$F^{(n)}(z) = \int_0^\infty e^{-u} u^n f^{(n)}(uz)\, du = \omega^{n-1} \int_0^\infty e^{-u} u^n f(u\omega^n z)\, du$$

since $f^{(n)}(t) = \omega^{n-1} f(\omega^n t)$. Since $L^{(n)}(1/z) = (-1)^n z^{n+1} \int_0^\infty u^n e^{-u} f(uz)\, du$, we have

$$F^{(n)}(z) = (-1)^n \omega^{-n^2-1} z^{-n-1} L^{(n)}(\omega^{-n}/z),$$

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