GENERALIZED LAPLACIANS OF HIGHER ORDER

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The generalized Laplacians of Blaschke and Privaloff can be extended to higher orders in the following way:

$$\nabla_p^{(n)} f(P) = \lim_{r \to 0} \frac{4}{r^2} [L(\nabla_p^{(n-1)} f; P; r) - \nabla_p^{(n-1)} f(P)],$$

$$\nabla_a^{(n)} f(P) = \lim_{r \to 0} \frac{8}{r^2} [A(\nabla_a^{(n-1)} f; P; r) - \nabla_a^{(n-1)} f(P)],$$

where $\nabla_p^{(0)} f(P) = \nabla_a^{(0)} f(P) = f(P)$, and $L(f; P; r)$, $A(f; P; r)$ are the mean values of $f(P) \equiv f(x, y)$ on the perimeter and on the interior, respectively, of a circle of center P and radius r. For $n = 1$ these equations give the definitions of first order generalized Laplacians as given by Blaschke [1] and Privaloff [3]. The purpose of this paper is to present some results on the operators of higher order.

We give as lemmas some known results concerning the operators $\nabla_p^{(1)} \equiv \nabla_p$, $\nabla_a^{(1)} \equiv \nabla_a$.

Lemma 1. (See [1], [3].) *If $f(P)$ has continuous second partial derivatives at P, then $\nabla_p f(P)$, $\nabla_a f(P)$ exist, and*

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \equiv \nabla^2 f(P) = \nabla_p f(P) = \nabla_a f(P).$$

Lemma 2. (See [1], [3].) *If $f(P)$ is continuous and $\nabla_a f = 0$ or $\nabla_p f = 0$ everywhere on an open domain $\mathfrak{D}$, then $f(P)$ is harmonic on $\mathfrak{D}$.*

Lemma 3. (See [4], [5].) *If $u(P)$ is a logarithmic potential function*

$$u(P) = \int_W \log \frac{1}{PQ} \, d\mu(Q),$$

where μ is a mass distribution with density $D_s\mu(P)$ defined on a domain W, then $\nabla_p u(P)$, $\nabla_a u(P)$ exist whenever $D_s\mu(P)$ exists, and $\nabla_p u(P) = \nabla_a u(P) = -2\pi D_s \mu(P)$.

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