DUALITIES OF FINITE PROJECTIVE PLANES

By Richard William Ball

Introduction. It is the object of this paper to study a very general type of transformation of a projective plane into itself, the duality (called by other authors auto-dualities, correlations, *etc.*). A duality δ of a projective plane is a one-to-one transformation of the set of all points (lines) of the plane into the set of all lines (points) of the plane with the property that point P is on line h if, and only if, point h^δ is on line P^δ.

In the study of point-to-point transformations one makes a special examination of the fixed elements of the transformation. The parallel concept in this paper is that of an absolute element. A point P is an absolute point of the duality δ if it is on its corresponding line P^δ. It is proved in §2 that every duality of a finite projective plane possesses at least one absolute point.

This and other properties of dualities of finite projective planes are derived from number-theoretical considerations. Since a finite projective plane can be thought of as an array of numbers (as treated in the study of Latin squares and other statistical arrays), it is entirely natural that one seek to characterize such a plane by number-theoretical properties. The most important of these is the existence of an integer $n \geq 2$ such that the plane contains exactly $n^2 + n + 1$ points and such that every line contains exactly $n + 1$ points (see [5]). The number n will retain this significance throughout the paper. Apart from finiteness, no assumption will be made on the nature of the plane. In particular the theorem of Desargues need not be valid in the plane.

In this paper we consider a duality and its powers simultaneously. Odd powers of a duality are again dualities and even powers are projectivities. In particular a duality of period 2 is called a polarity and a projectivity of period 2 an involution (see [6], especially page 268). In §2 we derive congruences between the numbers of absolute or fixed points of various powers of a duality. These are obtained by counting certain configurations in the plane which are associated with a duality and its powers.

We state in §3 some unpublished number-theoretical results due to Professor Ivan Niven. These enable us to prove the principal results in this paper.

Let $2o(\delta^2)$ be the (finite) order of the duality δ. Let $R(2o(\delta^2))$ be the group of residue classes prime to $2o(\delta^2)$. Let $N(\delta^i)$ be the number of absolute points of the duality δ^i. And let $n = n^*n'$ where n^* is the largest square dividing n. If n is a square, $N(\delta^i) = N(\delta)$ for all i in $R(2o(\delta^2))$. If n is not a square, it is shown that any one of the following is sufficient to prove that $N(\delta^i) = n + 1$ for all i in $R(2o(\delta^2))$:

1. n' does not divide $o(\delta^2)$.

Received May 24, 1948; in revised form July 2, 1948.