DIOPHANTINE SYSTEMS SUGGESTED BY BHASCARA'S PROBLEM

By E. Rosenthall

1. Bhascara's problem [4] is to find all rational integer pairs whose difference is a square and the sum of whose squares is a cube; this is essentially the problem of solving in rational integers the diophantine system

$$x^2 + y^2 = z^3, \qquad x - y = t^2. \tag{1}$$

Two sets of parametric solutions for this system have been given by Potts [4]. To obtain these solutions assumptions were made which were sufficient but not necessary ones, and though these parametric representations yield an infinity of solutions they may not give them all.

The complete solution of (1_1) is well known, but it cannot be handled with facility in imposing condition (1_2). If the additional condition $(x, y) = 1$ is imposed on (1) the problem is completely transformed and we obtain a diophantine system which by elementary means can be solved completely in integral formulas, that is we obtain each of x, y expressed by a set of polynomials in integral parameters having integral coefficients. This is stated in the following theorem.

Theorem 1. *All rational integers satisfying the system*

$$x^2 + y^2 = z^3, \qquad x - y = t^2 \qquad ((x, y) = 1) \tag{2}$$

are given, without duplication, by

$$x = r^3 - 3rs^2, \qquad y = 3r^2s - s^3 \tag{3}$$

where r and s are given by the following three sets of formulas:

$$r = p^4 + 6p^2q^2 - 8pq^3 + 21q^4, \qquad s = 4q(p^3 + q^3 - 3p^2q - 3pq^4); \tag{4}$$

$$r = 4p^4 + 8p^3q + 12p^2q^2 + 4pq^3 + 5q^4, \qquad s = 4q(2p^3 - 3pq^2 - q^3); \tag{5}$$

$$\begin{aligned} r &= -2(p^4 + 7q^4 - 2p^3q - 2pq^3), \\ s &= -[p^4 - 32pq^3 + 30p^2q^2 - 8p^3q + 13q^4]; \end{aligned} \tag{6}$$

and in each case r, s can be interchanged. The parameters p and q must be selected so that $(p, q) = 1$, $(p + q, 3) = 1$, $p > 0$; further in (4) and (6) p and q must be of opposite parity, but in (5) q is odd.

The resolution of this system is made possible by the following two results; the first can be found in [6; 393], and the second is due to Dickson [1; 44–47].

Received May 31, 1948.