THE THEORY OF LINEAR DIFFERENTIAL SYSTEMS BASED UPON A NEW DEFINITION OF THE ADJOINT

By G. J. Haltiner

0. **Introduction.** This investigation considers the formal aspects of the theory of an ordinary differential boundary system and of the associated theory of the expansion of an "arbitrary" function, the discussion being based upon a new definition of the adjoint recently given by R. E. Langer [11], [12], [13].

The differential system to be studied is expressible in the form

$$L(u, \lambda) = 0, \qquad A_i(u, \lambda) = 0 \qquad (i = 1, 2, \cdots, n), \tag{0.1}$$

where $L(u, \lambda)$ designates the linear operator of order n

$$L(u, \lambda) = \sum_{j=0}^{n} p_j(x)u^{(n-j)}(x) + \lambda q(x)u(x) \qquad (p_0(x) \equiv 1), \tag{0.2}$$

superscripts in parentheses denoting derivatives of the indicated order; and the boundary forms $A_i(u, \lambda)$ are linear, homogeneous functions of the values of $u(x)$ and its derivatives evaluated at two points, $x = a$ and $x = b$. Thus

$$A_i(u, \lambda) = \sum_{j=1}^{n} \alpha_{ij}(\lambda)u^{(n-j)}(a, \lambda) + \sum_{j=n+1}^{2n} \alpha_{ij}(\lambda)u^{(2n-j)}(b, \lambda), \tag{0.3}$$

$i = 1, 2, \cdots, n$. The coefficients $p_j(x)$ and $q(x)$ are to be differentiable functions of x on the closed interval (a, b), λ is to be a complex parameter, and the functions $\alpha_{ij}(\lambda)$ are to be taken to be polynomials of arbitrary degree in λ. This restriction to polynomial coefficients is one that could, in some respects, be relaxed. To insure the linear independence of the boundary relations, the $n \times 2n$ matrix $(\alpha_{ij}(\lambda))$ is to be assumed to be of rank n for every value of λ.

In this discussion the adjoint differential equation to be used will be the familiar one; however, a variation from the classical definition of the adjoint system will appear in the boundary relations. In the customary treatments of the theory of differential systems (see [2], [4], [5]) these boundary conditions are defined more or less indirectly.

The procedure calls for the selection of n linear forms A_i, $i = n + 1, \cdots, 2n$, of the type in (0.3), but such that the complete set A_i, $i = 1, 2, \cdots, 2n$, when considered as forms in $u(a), \cdots, u^{(n-1)}(a), u(b), \cdots, u^{(n-1)}(b)$, is of non-vanishing determinant. With this selection any linear combination of the elements $u(a), \cdots, u^{(n-1)}(a), u(b), \cdots, u^{(n-1)}(b)$, and in particular the elements themselves, will be expressible in terms of the quantities A_i.

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