INTEGRAL VALUED ENTIRE FUNCTIONS

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1. **Introduction.** Let I be the set of rational integers. An entire function $f(z)$ is said to be integral valued if $f(n) \varepsilon I$ for each n, $n = 0, 1, \cdots$. The class of all such functions forms a ring R. Let R_0 be the subring of R consisting of the integral valued polynomials; there are simple functions—*e.g.*, 2^z—which belong to R but not to R_0. More generally, any set of conjugate algebraic integers determines such functions. Let $Q(x) = 1 + q_1x + \cdots + q_mx^m = \prod_1^m (1 - \beta_kx)$ be irreducible and have integral coefficients. Since $Q(0) = 1$, $1/Q(x)$ has a formal power series with integral coefficients. If P is a polynomial with integral coefficients, then $P(x)/Q(x)$ has a formal expansion $\sum_0^\infty b_nx^n$ where $b_n \varepsilon I$ for all n. Expressing these coefficients in terms of the algebraic integers β_k, we have $b_n = \sum_1^m A_k\beta_k^n$ where the A_k are certain complex numbers. The function $f(z) = \sum_1^m A_k\beta_k^z$ is then an integral valued function. Over R_0, these functions generate a ring $R_1 \subset R$. The general member of R_1 has the form

$$f(z) = P_0(z) + \sum_1^r P_k(z)\gamma_k^z$$

where $P_0, \cdots, P_r$ are polynomials (not necessarily in R_0) and the γ_k form one or more complete sets of conjugate algebraic integers. For brevity, we shall refer to such a function as having $\gamma_1^z, \gamma_2^z, \cdots, \gamma_r^z$ as generators. We observe that any function in R_1 is an entire function of exponential type; this class we denote as usual by K. Let K_1 be a subclass of K. The general problem with which this paper deals is that of characterizing the functions belonging to $R \cap K_1$.

We adopt the notation of previous papers [2]: $h(\theta, f)$ is the growth function for the function f, and is defined as lim sup $r^{-1} \log |f(re^{i\theta})|$; $K[H(\theta)]$ is the class of functions in K with $h(\theta, f) \leq H(\theta)$, for all θ for which $H(\theta)$ is defined; $K(a, c)$ is the class of f in K such that $h(0, f) \leq a$, $h(\pi, f) \leq a$, and $h(\pm \pi/2, f) \leq c$.

We first observe that the problem of characterizing $K_1 \cap R$ fails to be meaningful if K_1 is too large. Thus, $g(z) \sin \pi z$ is integral valued for any $g \varepsilon K$. We therefore restrict K_1 to be a proper subclass of the class $K(a, \pi)$, the critical uniqueness class associated with the functionals $\{T_n\}$ where $T_n(f) = f(n)$. See [2]. In the language used there, we seek to find what sequences of integers are admissible for $\{T_n\}$ and K_1.

The first result of this nature was established by Pólya, and sharpened by Hardy. (See [17], [10], [11].) Roughly speaking, 2^z is the integral valued function of slowest growth which is not a polynomial; more precisely, if $f \varepsilon R \cap$

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