LIMITS FOR THE CHARACTERISTIC ROOTS OF A MATRIX. III

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This paper is a continuation of my papers [1] and [2]. The numeration of the theorems and equations will be continued.

The following theorem will be proved.

THEOREM 19. *Let $A = (a_{\kappa\lambda})$ be a square matrix of order n and $f_1(y), f_2(y), \cdots, f_n(y)$ be arbitrary polynomials. Denote the elements of the matrix $f_\nu(A)$ by $a_{\kappa\lambda}^{(f_\nu)}$ and set*

$$\sum_{\substack{\lambda=1\\ \lambda\neq\kappa}}^{n} | a_{\kappa\lambda}^{(f_\nu)} | = P_\kappa^{(f_\nu)} \qquad (\kappa, \nu = 1, 2, \cdots, n).$$

Each characteristic root ω of A satisfies at least one of the n inequalities

$$| f_r(\omega) - a_{rr}^{(f_r)} | \leq P_r^{(f_r)} \qquad (r = 1, 2, \cdots, n),$$

and at least one of the $n(n - 1)/2$ inequalities

$$| f_r(\omega) - a_{rr}^{(f_r)} | \, | f_s(\omega) - a_{ss}^{(f_s)} | \leq P_r^{(f_r)} P_s^{(f_s)} \qquad (r, s = 1, 2, \cdots, n; r \neq s).$$

The Theorems 1 and 11 are the special case $f_1 = f_2 = \cdots = f_n = y$. The more general case that the polynomials are equal, but not linear, follows at once from the fact that $f_r(\omega)$ is a characteristic root of $f_r(A)$. But it is of importance that we may choose a suitable polynomial for each row of a given matrix in order to obtain sharp bounds for the characteristic roots.

Often it will be sufficient to use only quadratic polynomials $y^2 - t_r y$ with arbitrary coefficients t_r for $f_r(y)$. For instance, let ω be the greatest in absolute value of the characteristic roots of the matrix

$$A = \begin{pmatrix} 9 & 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 \end{pmatrix}.$$

It will be shown by suitable choice of $t_1, t_2, \cdots, t_5$ that $9.061 < \omega < 9.215$. Actually we have $9.187 < \omega < 9.188$. Hence the error for the upper bound is approximately only .3%.

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