EULER'S PROBLEM ON SUMS OF THREE FOURTH POWERS

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I. Introduction.

1°. I prove in this paper the following result first stated elsewhere [1]: *If any non-trivial solution of Euler's diophantine equation*

$$(1.1) \qquad x^4 + y^4 + z^4 = w^4$$

in positive integers exists, then w must exceed ten thousand. ([1] gives references to the history of this well-known problem.)

The outline of the proof is as follows. We may clearly confine ourselves to co-prime solutions of (1.1), so that

$$(1.2) \qquad (x, y, z, w) = 1.$$

Here (x, y, z, w) stands as usual for the greatest common divisor of the integers x, y, z and w.

On considering (1.1) modulo 4, we see that w must be odd, and precisely one of x, y, z, odd. We shall assume throughout the proof that

$$(1.3) \qquad w \text{ and } z \text{ are odd}; \qquad x \text{ and } y \text{ are even}.$$

We shall prove the following two theorems in §II:

Theorem 1. *If* (1.1) *has a non-trivial solution in integers satisfying the conditions* (1.2) *and* (1.3), *then*:

$$(1.4) \qquad x \equiv y \equiv 0 \pmod 8; \qquad w \equiv 1 \pmod 8;$$

$$(1.5) \qquad \text{one of } w \pm z \equiv 0 \pmod{1024}.$$

Theorem 2. *Under the hypotheses of Theorem* 1,

$$(1.6) \qquad w \pm z = 2^{4\sigma+10+\epsilon} d^4 k, \qquad w \mp z = 2e^4 l;$$

$$(1.7) \qquad x = 2^{3+\sigma} deu, \qquad y = 2^{3+\sigma} dev,$$

where:

(1.8)
(i) $\sigma \geq 0$; $\epsilon = 0$ *unless u, v are both odd, when* $\epsilon = 1$;
(ii) d, e, k, l *are odd integers; not both u, v even*;
(iii) *Every prime factor of k and l is of the form* $8n + 1$;
(iv) $(d, e) = (d, l) = (e, k) = (u, v, dekl) = 1$.

2°. We see from formula (1.6) of Theorem 2 that

$$(2.1) \qquad w = 2^{4\sigma+9+\epsilon} d^4 k + e^4 l.$$

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