INEQUALITIES CONCERNING POLYGONS AND POLYHEDRA

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1. In 1943 at the mathematical competition of the "Loránd Eötvös Mathematical and Physical Society", Professor L. Fejér proposed the proof of the following fact: If R and r denote the radii of two concentric circles, the first containing, the second contained in, an acute-angled triangle Δ, then

$$R/r \geq 2. \tag{1}$$

For, decompose Δ into six right triangles by connecting the common center O of the circles with the vertices A, B, C of Δ and drawing perpendiculars from O to the sides. The sum of the six angles at O of these triangles being 2π, there is a triangle, say OAC', such that angle $AOC' \geq 2\pi/6$. Then we have $R \geq OA \geq 2OC' \geq 2r$. Equality holds only if the angles at O of all the six triangles equal $\pi/3$, *i.e.*, if Δ is an equilateral triangle with its center at O. (For all that is contained in §1, I am indebted to the friendly communication of Professor L. Fejér.)

2. The reason for which the elementary remark proved just now deserves attention is the fact that it can be considered as the common source of different theorems. As an immediate generalization of (1) let us mention, for example, that the inequality holds also without restriction to acute-angled triangles and—which is more important—without restriction to concentric circles. This result was obtained by L. Fejér in 1897 as a competitor at the competition mentioned above. (See [6], [9].) More generally

$$R_n/r_n \geq \sec \pi/n, \tag{2}$$

denoting by R_n and r_n the radii of two circles containing and contained in an arbitrary convex n-gon [1].

We shall need the following further generalization of (1): If R_1, R_2, R_3 are the distances of an arbitrary point O of the space from the vertices of a triangle of area t, then

$$A(R) \geq \tfrac{2}{3}(3^{\frac{1}{2}}t)^{\frac{1}{2}}, \tag{3}$$

denoting by $A(R)$ the arithmetic mean of the R_i's.

Let us agree at once upon the notation $A(x; k)$ and $H(x; k)$ for the arithmetic and harmonic mean of certain numbers x_i with the weights k_i, and upon the notation $A(x)$, $H(x)$ in the case $k_i \equiv 1$.

Making use of the well-known inequality $3^{\frac{1}{2}}t \geq 9r^2$, r being the radius of the inscribed circle, we obtain the inequality, found by M. Schreiber [10],

$$A(R) \geq 2r, \tag{4}$$

from which (1) follows under the more general conditions of (2).

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