FUNCTION CLASSES INVARIANT UNDER THE FOURIER TRANSFORM

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This note is a contribution to the following problem: Find function classes characterized by "simple" conditions which are left invariant by Fourier transforms. In a previous note [1] the writer found two such invariant classes defined by Möbius series. In the present note two more classes are exhibited: The first is characterized essentially by the condition $(-xd/dx)^n f(x) \geq 0$, $n = 0, 1, \cdots$, and the second by the condition $(-d/xdx)^n f(x) \geq 0$.

The J_1 class of functions $f(x)$ is defined by the conditions:

(a) $(-xd/dx)^n f(x) \geq 0$ $\quad (0 < x < \infty; n = 0, 1, \cdots)$;

(b) $f(x) \to 0$ $\quad (x \to +\infty)$;

(c) $xf(x) \to 0$ $\quad (x \to +0)$.

Let $q(s) = f(e^s)$. Then $(-d/ds)^n q(s) \geq 0$. Such functions have been termed *completely monotonic.* A fundamental theorem, due essentially to Hausdorff, states that the class of functions completely monotonic in the interval $0 < s < \infty$ is identical with the class $q(s)$ represented by the Laplace-Stieltjes integral

$$q(s) = \int_0^\infty e^{-st}\, d\rho(t) \qquad (s > 0), \tag{1}$$

where $\rho(t)$ is a non-decreasing function.

Theorem 1. *The class J_1 is identical with the class of the form*

$$f(x) = \int_0^1 x^{-t}\, d\rho(t) \qquad (x > 0), \tag{2}$$

where $\rho(t)$ is a non-decreasing function continuous at 0 *and* 1.

Proof. Relation (1) demands

$$f(x) = \int_0^\infty x^{-t}\, d\rho(t) \qquad (x > 1). \tag{3}$$

Obviously if $f(x)$ is contained in J_1, so also is $f(cx)$ for c any positive constant; thus, $f(cx) = \int_0^\infty x^{-t}\, d\rho_c(t)$, $x > 1$. If $u = cx$, then $f(u) = \int_0^\infty u^{-t}\, dr_c(t)$, $u > c$, where $r_c(t) = \int_0^t c^t\, d\rho_c(t)$. If c and b are positive constants less than one, then $\int_0^\infty e^{-st}\, dr_c = \int_0^\infty e^{-st}\, dr_b$ for $s \geq 0$. Hence, by the uniqueness theorem of the Laplace integral, $r_c(t)$ and $r_b(t)$ are substantially equal. It follows that (3) is actually valid for all $x > 0$.

Condition (b) obviously is equivalent to $\rho(t)$ being continuous at $t = 0$. To

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