THE SUPPORT PROPERTY OF A CONVEX SET IN A LINEAR NORMED SPACE

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1. **Introduction.** In a linear normed space S, each translate of a linear proper subset of S is called a linear variety, and a maximal closed linear variety is called a *hyperplane*. That is, the linear variety H is a hyperplane if there is no closed linear variety L such that $H \subset L \subset S$ and $H \neq L \neq S$. If H is a hyperplane and X a subset of S, X is said to *lie on one side of* H if each line segment joining two points of $X \sim H$ is disjoint from H. (We will reserve the minus sign "$-$" for linear differences, and the plus sign "$+$" for linear sums. Set differences and sums will be denoted respectively by "$\sim$" and "$\cup$", set products by "$\cap$".) If X lies on one side of H and the distance between X and H is zero, H is said to be a *plane of support* of X. If in addition H contains the boundary point p of X, X is said to be supported at p by H.

These definitions may be stated equivalently in terms of linear functionals on S. Thus each hyperplane can be characterized for some linear functional $f \not\equiv 0$ on S and some constant c as the set $[f; c]$ of all points x in S for which $f(x) = c$. X lies on one side of $[f; c]$ if $f(x) \geq c$ for each x in X or $f(x) \leq c$ for each x in X. X is supported at p by $[f; c]$ if X lies on one side of $[f; c]$ and $f(p) = c$.

It is well known that a convex set in Euclidean n-space is supported at each of its boundary points. (The simplest proofs of this fact are those by Botts and McShane in [2].) As we show by an example in §4 of this paper, this statement does not necessarily hold in an arbitrary linear normed space. (Another example is given in [4].) In fact, in each space l^p for $p \geq 1$ there is a closed convex set which is supported only at each point of a set of first category in its boundary. We are able, however, to establish the results indicated below in Theorems 1–3.

THEOREM 1. *If C is a closed convex cone and $C \neq S$, then C has at least one plane of support.*

THEOREM 2. *If C is convex and has an interior point, then C is supported at each of its boundary points.*

THEOREM 3. *If C is closed and convex and satisfies any one of the following conditions, C is supported at each point of a set dense in its boundary.*

(i) *Bounded sets in C are weakly compact;*

(ii) *S is an adjoint space and bounded sets in C are weakly compact as sets of functionals;*

(iii) *S is an adjoint space and C is transfinitely closed in S.*

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