A SPECIALIZATION OF ZORN'S LEMMA

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1. **An induction method.** By a partially ordered system $(A, >)$ we mean a set A and a transitive relation $>$. An element $a \varepsilon A$ is an upper bound of a subset B of A if $a \geq b$ for every b in B. An element $a \varepsilon A$ is maximal if $b \geq a$ implies $a \geq b$ for all b in A.

Zorn's lemma may be stated in the following form:

If in a partially ordered system $(A, >)$ each simply ordered subset has an upper bound in the system, then there exists at least one maximal element $a \varepsilon A$, with $a \geq a_0$ for a preassigned a_0.

Difficulties encountered in applying Zorn's lemma usually arise in proving that the simply ordered subsets have upper bounds. We shall show that for an important class of partially ordered systems it is possible to replace the phrase "simply ordered subset" in Zorn's lemma by "ascending sequence".

We say that an ascending sequence $a_1 \leq a_2 \leq a_3 \leq \cdots$ is cofinal in a simply ordered set L if $a_i \varepsilon L$ for each positive integer i and if corresponding to each $a \varepsilon L$ there is some a_i such that $a_i \geq a$. If each simply ordered set of a partially ordered system contains a cofinal ascending sequence, we say that the partially ordered system is of type ω. The following theorem is an immediate consequence of Zorn's lemma.

Theorem A. *If in a partially ordered system $(A, >)$ of type ω each ascending sequence has an upper bound in the system, then there exists at least one maximal element $a \varepsilon A$, with $a \geq a_0$ for a preassigned a_0.*

The next theorem is useful in verifying that certain partially ordered systems are of type ω. In this connection, see [3].

Theorem B. *If $(A, >)$ is a partially ordered system which has the property that corresponding to each simply ordered set L of the system there is a real-valued function f on L such that $f(a) > f(b)$ if and only if $a > b$, then $(A, >)$ is of type ω.*

Proof. Let L be simply ordered and let f be the corresponding real-valued function on L. Let s be the supremum of the set $f(L)$ of real numbers. If $s \varepsilon f(L)$, define $x_i = s$ for each positive integer i. Otherwise choose a sequence of numbers $x_1 < x_2 < x_3 < \cdots$ in $f(L)$ such that $\lim_{i\to\infty} x_i = s$. The sequence $f^{-1}(x_1), f^{-1}(x_2), \cdots$ is an ascending sequence which is easily seen to be cofinal in L. It follows from the definition that $(A, >)$ is of type ω.

2. **Applications.** We now give new proofs of some well-known theorems about completely additive set functions. For the definitions of terms used in this section, see [4].

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