OPERATIONS IN LINEAR METRIC SPACES

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A theorem of S. Banach [1; 234–238] concerning the inversion of a linear operation states that a linear (*i.e.*, additive and continuous) one-to-one operation $y = Sx$ transforming a Banach space X on another Banach space Y, possesses a linear inverse $x = S^{-1}y$. J. Schauder (see [4; 302, Erster Hauptsatz] and [8]) extended the theorem to not necessarily one-to-one transformations and S. Banach [2; 38–40] to the spaces of type (F).

A point set X is a linear metric space, if, for the elements x, y of X and real numbers a, sums $x + y$ and products ax are defined obeying the rules of the linear vector algebra, if X is a metric space and if the operations $x + y$ and ax are continuous in the metric of X. A linear metric space X is a space of type (F), if X is complete and if

$$\rho(x + z, y + z) = \rho(x, y) \tag{1}$$

holds for any $x, y, z \in X$. It is known [2] that a linear image $Y = SX$ of a space X of type (F) is either complete or of the first category in itself.

In this note we prove the theorem on the inversion of linear operations for arbitrary linear metric spaces, for which (1) need not be true (see §1, Theorems 1 and 2). Thus a conjecture of S. Mazur [2; 232] is confirmed. For this purpose the lemma of §1 as well as theorems of Banach, Sierpinski and Nikodym concerning sets with Baire property are essential. A linear image $Y = SX$ of a linear metric space X can well be of the second category in itself without being complete. Therefore, we are compelled to formulate our theorems in a somewhat weaker form than the respective theorems for the spaces of the type (F).

As another application of the lemma of §1 sequences of linear operations are treated in §3. We thus arrive at theorems, which, for spaces of type (F), were given by S. Mazur and W. Orlicz [6].

The last paragraph (§4) is dedicated to a problem of S. Banach [2; 232]. A complete linear metric space X being given, is there a new metric in X equivalent to the given one, which transforms X into a space of type (F)? This problem being as yet unsolved, we give necessary and sufficient conditions for the existence of the metric required.

1. A lemma.

Lemma. *Let Y be a linear metric space and let $\Phi(y)$ be a real-valued function defined in Y, having the Baire property* (see [5; §45]) *and satisfying the following conditions:*

(a) $\overline{\lim}_{n\to\infty} \Phi(a_n y) \leq 0$ *for* $a_n \to 0$ *and every* $y \in Y$;

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