INEQUALITIES FOR THE CAPACITY OF A LENS

By John G. Herriot

1. **Introduction.** A lens may be described simply as a solid determined by the intersection of two spheres. More precisely, if $c > 0$, the solid of revolution generated by revolving about the imaginary axis the area in the complex z-plane defined by the inequalities

$$\theta_1 \leq \arg \frac{z - c}{z + c} \leq \theta_2$$

is called a lens. We may suppose $0 < \theta_1 \leq \theta_2 < 2\pi$. It is, however, more convenient to characterize a lens in terms of its exterior angles. Accordingly we denote by α and β the exterior angles which the two portions of the boundary of the generating area make with the real axis. It is easily seen that $\beta = \theta_1$, $\alpha = 2\pi - \theta_2$. We shall assume, as we may without loss of generality, that $\alpha \leq \beta$. The sum of these angles $\alpha + \beta$ is called the dielectric angle of the lens. Clearly $\alpha + \beta \leq 2\pi$ and hence we need consider only values of α not exceeding π. Sometimes it is convenient to introduce the radii a and b of the intersecting spheres; these are given by $c = a \sin \alpha = b \mid \sin \beta \mid$.

It is clear that when $\alpha + \beta = \pi$ the lens becomes a sphere and when $\alpha + \beta \geq \pi$, $\beta \leq \pi$ the lens is convex. When $\beta \neq 0$ and $\alpha \to 0$, keeping a fixed, the lens becomes a sphere of radius a. When α, $\beta \to 0$ in such a manner that $\beta = k\alpha$, and a is kept fixed, the lens becomes two tangent spheres of radii a and a/k. When α, $\beta \to \pi$, keeping c fixed, the lens becomes a circular disk of radius c.

The electrostatic capacity C of a conducting solid is simply the electrostatic charge required to produce a unit potential on the surface of the solid. (For other equivalent definitions of capacity see G. Pólya [3], G. Pólya and G. Szegö [5], [6], G. Szegö [8], [9], [10].) We wish to compare the capacity of the lens with other quantities more easily determined. The volume radius V^* of a solid is the radius of the sphere having the same volume as the solid. The surface radius S^* is the radius of the sphere having the same surface area. The mean radius M^* of a convex solid is the radius of the sphere with the same integral of mean curvature as the solid. Note that M^* is defined only for convex solids. For solids of revolution we consider also the outer radius r^* of the meridian section of the solid, the meridian section being the closed curve in which a plane containing the axis of revolution intersects the solid. The outer radius r^* of the meridian section can be defined as the radius of the uniquely determined circle onto the exterior of which the exterior of the meridian section can

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