A CLASS OF GAP THEOREMS

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Let $f(z) = \sum_0^\infty a_n z^n$, with $|z| = 1$ as circle of convergence. The following theorem was recently proved by Ilieff [6].

Theorem 1. *If the coefficients* a_n *are bounded and if there are increasing sequences* $\{n_p\}$ *and* $\{m_p\}$ *such that* $\liminf_{p\to\infty} |a_{n_p}| > 0$, $\lim_{p\to\infty} |a_{l+n_p}| = 0$ $(l = 1, 2, \cdots, m_p)$, *then* $f(z)$ *cannot be continued analytically beyond* $|z| = 1$.

In other words, the unit circle is a natural boundary if the sequence of coefficients is bounded and has an infinite number of gaps, with unbounded lengths, while the coefficient immediately preceding each gap does not approach zero. This result includes a theorem of Erdös and Piranian [5; 657].

I shall give a new proof of Theorem 1, using an idea of Duffin and Schaeffer [4], which at the same time leads to results which are more general in several directions. While this paper was in the course of publication, still more general results were announced by Agmon [0].

The observation that the circle of convergence is a natural boundary if the sequence of coefficients has long gaps in which the coefficients are small, while the coefficient just preceding each gap is as large as possible, was also made by Lösch [7] and further developed by Claus [2]. Both these authors assume gaps whose lengths are connected with the magnitude of the a_n, the gaps being longer if the coefficients are allowed to be larger. I shall generalize Theorem 1 to deal with power series whose coefficients are not necessarily bounded; the generalized theorem does not contain the results of Lösch and Claus, but it suggests strongly that their hypotheses on the length of the gaps are unnecessary and that it is always sufficient to have gaps whose lengths become infinite. The "little" Fabry gap theorem can also be regarded as a theorem of the same kind; it states that the unit circle is a natural boundary if $a_n = 0$ except for $n = n_p$, where $n_{p+1} - n_p \to \infty$; here, since only the coefficients at the ends of the gaps are different from zero, the largest coefficients are automatically among them.

Theorem 1 is contained in the following result, in which we consider only the coefficients whose indices belong to an arithmetic progression. To simplify the notation, we write $a(n)$ instead of a_n.

Theorem 2. *Let* β *and* γ *be positive integers; let* $\{n_p\}$ *and* $\{m_p\}$ *be (strictly) increasing sequences of positive integers. Let* $|a(n\beta + \gamma)| \le M$ $(n = 1, 2, \cdots)$, *let* $\liminf_{p\to\infty} |a(n_p\beta + \gamma)| > 0$ *and let* $\lim_{p\to\infty} a(n_p\beta + \gamma + l) = 0$, $l = 1, 2, \cdots, m_p$. *Then if* $\alpha = \pi(1 - \beta^{-1})$, *for no interval* $|\theta - \theta_0| < \delta$, $\delta > 2\alpha$, *can we have*

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