BASIC SETS OF POLYNOMIALS. I

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1. In J. M. Whittaker's terminology [10], [11], a basic set of polynomials $p_n(z)$ is a set such that every polynomial can be written as a finite linear combination of the $p_n(z)$. Among the various possible expansion problems associated with basic sets, two have received particular attention: (a) the problem of when every function analytic in a circle $|z| < R$ (or in $|z| \leq R$, respectively) can be expanded in a series of the $p_n(z)$, uniformly convergent in every circle $|z| \leq R' < R$ (or $|z| \leq R$, respectively); the set is called effective in $|z| < R$ (or in $|z| \leq R$, respectively) if this expansion is possible; (b) the problem of when every entire function of a specified class can be expanded in a series of the $p_n(z)$, uniformly convergent in every circle; I propose to say that the set is effective for the given class of entire functions in this case. In the first part of this note we shall show that every theorem of type (a) leads to a corresponding theorem of type (b) for an associated basic set obtained by a Laplace transform (or a generalization). As one application, a theorem on Laguerre polynomial expansions is obtained.

Since non-trivial basic series may represent zero, a basic polynomial expansion is not in general unique, and in Whittaker's terminology effectiveness demands representation by a basic series in which the coefficients are determined in a particular way. However, for expansions in a finite circle effectiveness is known [2], [11; 30] to imply uniqueness, so that the coefficients of the expansions obtained in problem (a) must coincide with those obtained by Whittaker's method. Since the expansions considered here in problem (b) are transforms of expansions obtained from problem (a), their coefficients also have the standard form, even though a given entire function may have many expansions in terms of a given basic set.

In the second part, problem (a) is considered for simple sets, that is, those for which $p_n(z)$ is of degree n. The problem has been discussed by Whittaker and others [4], [6], [10], [11], under various hypotheses on the coefficients or the zeros of the $p_n(z)$. Here we shall show that many of their criteria can be obtained from the principle that $p_n(z)$ is "nearly" z^n when $|z|$ is large. This in a sense explains why a set $\{p_n(z)\}$ is usually effective only in sufficiently large circles. By the use of the theorems of the first part, various criteria for the expansion of entire functions can be written down at once.

2. Let $\lambda_n = \lambda_n(\rho, \sigma)$ be a sequence of non-zero numbers such that $\lambda_n \sim (\sigma e\rho/n)^{-n/\rho}$ as $n \to \infty$, for example $\lambda_n = \Gamma(1 + n/\rho)\sigma^{-n/\rho}$. We prove the following theorem.

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