SETS OF COMPLEX NUMBERS ASSOCIATED WITH A MATRIX

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1. **Introduction.** Let $A = (a_{ij})$ be a square matrix of order n whose elements are complex numbers. If $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ are vectors such that

$$(1) \qquad x\bar{x}' = \sum_{i=1}^{n} x_i\bar{x}_i = 1, \qquad y\bar{y}' = \sum_{i=1}^{n} y_i\bar{y}_i = 1,$$

then $xA\bar{y}' = \sum_{ij=1}^{n} a_{ij}x_i\bar{y}_j = \alpha$, where α is a complex number. If S_1 is the set of all complex numbers of the form $xA\bar{y}'$ where x and y satisfy conditions (1), then S_1 is the set of all complex numbers in or on the circle of radius ρ_n about zero in the complex plane, where ρ_n^2 is the largest of the characteristic roots of $A\bar{A}'$ (see [3]). It is the purpose of this paper to investigate this set further and also to investigate two subsets of this set. The set S_1 is the set of elements of all matrices $UA\bar{V}'$ where U and V are unitary matrices $(U\bar{U}' = V\bar{V}' = I)$.

The set S_2 consisting of all complex numbers of the form $xA\bar{x}'$, where x satisfies (1), is a closed convex set in the complex plane and is called the *field of values of A* (see [1]). The set S_2 is the set of all diagonal elements of all matrices $UA\bar{U}'$ where U is a unitary matrix. Hence S_2 is unchanged if A is replaced by $UA\bar{U}'$. The set S_3 consisting of all complex numbers of the form $xA\bar{y}'$, where x and y satisfy (1) and also $x\bar{y}' = 0$, is the set of all non-diagonal elements of all matrices $UA\bar{U}'$ where U is a unitary matrix. The set S_3 is also unchanged if A is replaced by $UA\bar{U}'$.

2. **The sets S_2 and S_3.** If the characteristic roots of $A\bar{A}'$ are $\rho_1^2 \leq \rho_2^2 \leq \cdots \leq \rho_n^2$ and $R = \text{diag.}\{\rho_1, \rho_2, \cdots, \rho_n\}$ where $\rho_i \geq 0$, there exist unitary matrices U and V such that $\bar{U}'AV = R$ (see [2; 78]). Hence $UR\bar{V}' = A = (a_{ij})$ and $a_{ij} = u_iR\bar{v}_j'$, where $u_i = (u_{i1}, u_{i2}, \cdots, u_{in})$ and $v_i = (v_{i1}, v_{i2}, \cdots, v_{in})$ are the i-th rows of U and V respectively. Write $|u_{ik}| = \xi_{ik}$ and $|v_{ik}| = \eta_{ik}$ and it follows that

$$|a_{ij}| \leq \sum_{k=1}^{n} \rho_k\xi_{ik}\eta_{jk} \leq \tfrac{1}{2}\sum_{k=1}^{n} \rho_k(\xi_{ik}^2 + \eta_{jk}^2) \leq \tfrac{1}{2}\,\rho_n \sum_{k=1}^{n} (\xi_{ik}^2 + \eta_{jk}^2) = \rho_n,$$

since

$$\sum_{k=1}^{n} \xi_{ik}^2 = \sum_{k=1}^{n} \eta_{jk}^2 = 1.$$

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