DIFFERENTIAL EQUATIONS WITH NON-OSCILLATORY EIGENFUNCTIONS

By Philip Hartman

1. **The statement of the theorem.** In the differential equation

$$x'' + (\lambda - q)x = 0 \tag{1}$$

and boundary condition

$$\alpha x(0) + \beta x'(0) = 0 \qquad (\alpha^2 + \beta^2 \neq 0), \tag{2}$$

let $q = q(t)$, where $0 \leq t < \infty$, be a real-valued, continuous function; λ a real-valued eigenvalue parameter; and α, β real numbers. In the sequel, by a "solution of a differential equation" will be meant a real-valued, non-trivial ($\not\equiv 0$) solution. According to Weyl's classification [11; 238], (1) is in the *Grenzpunktfall* if (1) and (2) determine a non-degenerate eigenvalue problem, that is, if there is at least one solution $x = x(t)$ of (1) for which

$$\int_0^\infty x^2(t)\, dt < \infty \tag{3}$$

fails to hold. (Of course, since any solution is continuous for $t \geq 0$, only its behavior for large t is involved in the (L^2)-condition (3).) If there is some value of λ for which (1) has at least one solution $x = x(t)$ that does not satisfy (3), then the same is true for every λ (see Weyl [11; 238]; see also Wintner [14; 266, (iv bis)] and Bellman [1; 513]). A number λ is said to be an eigenvalue, if there exists an $x = x(t) \not\equiv 0$ satisfying (1), (2) and (3); the function $x = x(t)$ is called an eigenfunction belonging to λ.

When $q(t)$ satisfies the unilateral restriction

$$-\infty < \mu \leq \infty, \qquad \mu = \liminf_{t\to\infty} q(t), \tag{4}$$

Weyl [11; 251–257] has shown that (1) is in the Grenzpunktfall and that the set of points λ of the spectrum satisfying $\lambda < \mu$ is either empty or consists of a finite or infinite increasing sequence of eigenvalues $\lambda_0 < \lambda_1 < \cdots < \mu$. When $\mu < \infty$, the number n of such eigenvalues, $0 \leq n \leq \infty$, is the same as the number of zeros on $0 < t < \infty$ of a solution $y = y(t) \not\equiv 0$ of

$$y'' + (\mu - q)y = 0 \tag{5}$$

and

$$\alpha y(0) + \beta y'(0) = 0. \tag{6}$$

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