GENERALIZED INVERSION FORMULAS FOR CONVOLUTION TRANSFORMS

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1. Introduction. The authors have previously studied the class of convolution transforms

$$f(x) = \int_{-\infty}^{\infty} G(x - t)\phi(t)\, dt \tag{1}$$

for which the kernel $G(t)$ has a representation of the form

$$G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{E(s)}\, ds. \tag{2}$$

Here

$$E(s) = e^{bs} \prod_{k=1}^{\infty} \left(1 - \frac{s}{a_k}\right) e^{s/a_k} \tag{3}$$

where b, $\{a_k\}_1^{\infty}$ are real constants subject to the sole restriction that

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty. \tag{4}$$

See [2] and [3]. This theory includes as special cases the Laplace and Stieltjes transforms. If we set $E(s) = \cos \pi s$ then $G(t) = (\operatorname{sech} \frac{1}{2}t)/2\pi$ and the corresponding convolution transform (1) becomes, after a change of variables, the Stieltjes integral equation

$$F(y) = \int_{0+}^{\infty} \frac{\Phi(u)}{u + y}\, du. \tag{5}$$

Similarly if $E(s) = \Gamma(1 - s)$ then $G(t) = e^{-e^t} e^t$ and after a change of variables the corresponding convolution transform reduces to

$$F(y) = \int_{0+}^{\infty} e^{-uy} \Phi(u)\, du,$$

which is Laplace's integral equation.

In the previously mentioned papers the authors have determined the convergence behavior of the transform (1). A kernel $G(t)$ is said to belong to class I if there are both positive and negative a_k's; to class II if there are only positive a_k's and if $\sum_1^{\infty} 1/a_k = \infty$; and to class III if there are only positive a_k's and if $\sum_1^{\infty} 1/a_k < \infty$. Either $G(t)$ or $G(-t)$ belongs to one of these three classes. The authors proved, for example, that if $G(t) \in$ class I and if the transform (1) exists as a conditionally convergent integral for any single value

Received April 8, 1948.