TENSOR INVARIANTS OF LIE GROUPS

By H. C. Lee

Introduction. A Lie group induces two "parameter groups" in the group manifold, namely the left and right *translation groups*. By Cartan's exterior calculus it is easy to find all the exterior forms which are invariant under one of these translation groups (see [6; 65]). Hodge [5; 249–255], applying his theory of harmonic integrals, has given another method to determine these invariant forms (for compact semisimple groups only). But an exterior form is, essentially, a tensor of a very special kind, and the above methods are by their nature not applicable to the general case. We shall give here a direct method of finding all the invariant tensors of either or both of the two translation groups in the group manifold.

1. **Invariant tensors.** Consider an r-dimensional manifold of class ≥ 2, with local coordinates a^α, $\alpha = 1, \cdots, r$. (The letter a here used is intended to denote the parameters of a Lie group in subsequent consideration. We shall follow the long used notation that the parameters are denoted by a, b, c, and that they carry the indices α, β, γ, $\delta = 1, \cdots, r$. (See [4].)) Let ξ^α be a given contravariant vector whose components are functions of the a's of class ≥ 1. Then, if $T^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p}$ is a tensor of class ≥ 1, say for generality a relative tensor of weight W, the formation

$$(1.1)\quad \begin{aligned} &\xi^\gamma \frac{\partial}{\partial a^\gamma} T^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p} + \sum_{r=1}^{p} T^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_{r-1} \gamma \alpha_{r+1} \cdots \alpha_p} \frac{\partial \xi^\gamma}{\partial a^{\alpha_r}} \\ &\qquad - \sum_{s=1}^{q} T^{\beta_1 \cdots \beta_{s-1} \gamma \beta_{s+1} \cdots \beta_q}_{\alpha_1 \cdots \alpha_p} \frac{\partial \xi^{\beta_s}}{\partial a^\gamma} + W T^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p} \frac{\partial \xi^\gamma}{\partial a^\gamma} \end{aligned}$$

which we denote symbolically by $\Delta T^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p}$, is a tensor. (This can be seen for example by the introduction of a symmetric linear connection. Then (1.1) is equal to the same expression wherein ordinary differentiation is replaced throughout by covariant differentiation with respect to the linear connection. The summation convention is always understood for repeated indices.)

It can be verified [5; 240–241] that the condition $\Delta T = 0$ expresses the invariance of the tensor T under the infinitesimal transformation $\bar{a}^\alpha = a^\alpha + \epsilon \xi^\alpha$, and therefore under the group generated by it, that is, the one-parameter group whose *fundamental vector* is ξ^α.

For an r-parameter group with the fundamental vectors $\xi^\alpha_1, \cdots, \xi^\alpha_r$, a tensor T is invariant if $\Delta_1 T = 0, \cdots, \Delta_r T = 0$, where $\Delta_\lambda T$ denotes the expression (1.1) when we set $\xi^\alpha = \xi^\alpha_\lambda$, $\lambda = 1, \cdots, r$.

Received August 1, 1947; in revised form March 11, 1948.