KERNEL FUNCTIONS IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE

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Introduction. The theory of certain classes of partial differential equations of elliptic type may be developed by means of the concept of orthonormal systems of functions. Two important facts must be given in the definition of such systems:

(a) The metric with respect to which the orthonormalization is to be carried out.

(b) The class of functions from which the elements of the system are to be chosen.

For a large class of partial differential equations of elliptic type which are investigated in a given domain B, one may define in a most natural way a definite quadratic Dirichlet or energy integral, extended over the domain, which defines the metric to be used. It is an integral whose Euler-Lagrange equation (of the extremum problem related with it) coincides with the differential equation considered. This metric has been extensively used in the study of boundary value and eigenvalue problems connected with the equation [5; Chapter 7].

However, the class of functions on which this metric was applied was mostly the general class Ω of all differentiable functions for which the integral converges or a subclass Ω^0 of it defined by certain conditions on the boundary of the domain. Curiously enough, the important subclass Λ of Ω which consists of the solutions of the differential equation considered has been widely neglected. There is, however, an important paper of Zaremba [9] in which this subclass and an orthonormal system in it are studied in the special case of Laplace's equation, and where interesting results are obtained in this way. These investigations seem to have been the first along this line.

One important advantage of the study of orthonormal systems in the subclass Λ of the solutions was indicated in [2]. Namely, while in the wider systems as Ω and Ω^0 the bilinear kernel of the orthonormal system does not converge, it converges uniformly in every closed subdomain of B in the case of the class Λ. This fact makes the theory of the class Λ and its orthonormal systems much easier than that of the wider classes which had been preferred before. A close relation between the kernel function of the class Λ and important fundamental solutions of the differential equation was first established in the case of Laplace's equation [7]. These relations were generalized by us in two previous papers and the connection between kernel function, Green's and Neumann's function with respect to the differential equation were established [3], [4]. The practical and theoretical advantages of using this new concept were shown.

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