THE RADICAL OF A RING

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1. **Introduction.** In a recent paper [1], a radical N has been defined for an arbitrary ring R. It was shown that N is the classical radical if the right ideals of R satisfy the descending chain condition, that N coincides with the Jacobson radical [2] of R under this condition or if R is commutative, and that N always contains the Jacobson radical. The entire theory of N was based on transfinite methods through the early use of Zorn's Lemma. This was due partly to the fact that some of the theory was formulated as a special case of the more general F-radical theory, and partly to inability at that time to find proofs by finite methods.

In the present paper, no use is made of transfinite induction. Elementary proofs are given of certain theorems in [1], among them that the radical N is a two-sided ideal in R, that the residue class ring R/N has zero radical, that the radical of the matrix ring R_n is N_n. New results include a convenient characterization of the radical (Theorem 3), and a determination of the radical of an ideal R' in R as $N \cap R'$. It may be of interest to remark that with the exception of Theorem 3 all the theorems mentioned in this paragraph remain valid if N is replaced by the Jacobson radical.

The radical of a group algebra used by Segal [3] is precisely the radical N characterized as the intersection of the two-sided ideals M in R such that R/M is a simple ring with unit element [1; Theorem 7].

2. **Definitions and elementary properties.** The word *ideal* is to mean two-sided ideal.

If R is a ring and $a \in R$, we associate with the element a the ideal

$$G(a) = \{ar - r + \sum (x_i a y_i - x_i y_i)\}$$

in R, where r, x_i, y_i range over R and each summation is over some finite range. It follows that if $a \to \bar{a}$ is a homomorphism of R onto a ring $\bar{R}$, and $\overline{G(a)}$ denotes the homomorphic image in $\bar{R}$ of the ideal $G(a)$, then $G(\bar{a}) = \overline{G(a)}$.

Definition 1. An element a of R is *G-regular* if and only if $a \in G(a)$. An ideal in R is *G-regular* if and only if each element of the ideal is G-regular.

Definition 2. The *radical* N of a ring R is the set of elements b of R such that the principal ideal (b) is G-regular.

Thus $b \in N$ if and only if $a \in (b)$ implies that $a \in G(a)$.

Lemma 1. *If $a - c$ is G-regular and $c \in G(a)$, then a is G-regular.*

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