A NEW REPRESENTATION AND INVERSION THEORY FOR THE LAPLACE INTEGRAL

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1. **Introduction.** In this paper we will consider a new inversion formula for the Laplace transform. We will show that if

$$f(s) = \int_0^\infty e^{-st}\phi(t)\,dt, \tag{1}$$

then under certain rather severe restrictions on $\phi(t)$,

$$\phi(y) = \lim_{k\to\infty}\frac{1}{\Gamma(k)}\int_0^\infty (syk)^{k/2}J_k(2(syk)^{\frac{1}{2}})f(s)\,ds, \tag{2}$$

where J_k is the Bessel function of order k,

$$J_k(x) = (\tfrac{1}{2}x)^k\sum_{\nu=0}^\infty \frac{(-1)^\nu(\frac{1}{2}x)^{2\nu}}{\nu!\Gamma(\nu+k+1)}. \tag{3}$$

By suitable devices all restrictions on $\phi(t)$ may be removed, and our inversion formula made entirely general. Following Widder [8; 302–324] and Boas and Widder [1] we will construct from our inversion formula a fairly complete representation theory. From this theory we obtain a new proof of the Bernstein-Widder theorem on completely monotonic functions. We also obtain among other results, a new characterization of completely monotonic functions, a necessary and sufficient condition for a function $f(s)$ continuous and bounded $(0 \le s < \infty)$ to be completely monotonic being that its Fourier coefficients with respect to every generalized Laguerre polynomial of integral order be non-negative, *i.e.*,

$$\int_0^\infty e^{-s}s^kL_n^{(k)}(s)f(s)\,ds \ge 0 \qquad (n = 0, 1, 2, \cdots; k = 0, 1, 2, \cdots), \tag{4}$$

where

$$L_n^{(k)}(s) = \sum_{\nu=0}^n \binom{n+k}{n-\nu}\frac{(-s)^\nu}{\nu!}. \tag{5}$$

Let us discuss briefly the structure of our inversion formula. For the moment we proceed formally. We define

$$g(x) = \int_0^\infty J_0(2(sx)^{\frac{1}{2}})f(s)\,ds, \tag{6}$$

where, as before,

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