CHARACTERISTIC ROOTS AND THE FIELD OF VALUES OF A MATRIX

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1. **Introduction.** If A is a square matrix of order n whose elements are complex numbers, it is well known that the characteristic roots of A belong to the set of complex numbers $xA\bar{x}'$ where $x = (x_1, x_2, \cdots, x_n)$ is a vector such that $x\bar{x}' = \sum_{i=1}^{n} x_i\bar{x}_i = 1$. (The vectors used throughout this paper will be assumed to be of this type.) The set of all complex numbers $xA\bar{x}'$ is called the *field of values of* A (see [12]).

Since about 1900 several authors have obtained limits for the characteristic roots. Most of the earlier papers showed the characteristic roots to lie in certain circles about zero in the complex plane and were actually limits for the field of values of A. In an address before the Mathematical Association of America in 1938, Browne [6] gave a summary of these results up to that time.

In a recent paper, Brauer [2] has shown that each characteristic root lies in at least one of n circles having as centers the diagonal elements of A. In a second paper [3], he improved his results by showing that each characteristic root lies in at least one of $n(n - 1)/2$ Cassini ovals about the diagonal elements of A.

In this paper it will be shown that the field of values of A lies in a circle which is in general smaller than the ones previously given.

2. **The field of values of** A. If all the components of the vector x are zero except $x_{i_1}, x_{i_2}, \cdots, x_{i_r}$ and $\xi = (x_{i_1}, x_{i_2}, \cdots, x_{i_r})$, then $xA\bar{x}' = \xi M\bar{\xi}'$ where M is the principal sub-matrix containing the rows $i_1, i_2, \cdots, i_r$ of A. Hence the field of values of M is contained in the field of values of A.

Theorem 1. *The field of values of every principal sub-matrix of A is a sub-set of the field of values of A.*

It is well known that the field of values of $UA\bar{U}'$, where U is a unitary matrix $(U\bar{U}' = I)$, is identical with the field of values of A. It follows that every diagonal element of every unitary transform of A is in the field of values of A. If $\mu = u_iA\bar{u}_i'$ is an element of the field of values of A and U is a unitary matrix whose i-th row is the vector u_i, then the i-th diagonal element of $UA\bar{U}'$ is μ. Hence every element of the field of values of A is a diagonal element of some unitary transform of A.

Theorem 2. *The field of values of A is identical with the set of all diagonal elements of the unitary transforms of A.*

Write $A = F + iG$ where F and G are Hermitian. If A is normal $(A\bar{A}' =$

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