GRADIENT MAPPINGS AND EXTREMA IN BANACH SPACES

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1. **Introduction.** As is well known many boundary and eigenvalue problems lead to the consideration of completely continuous linear integral operators of the form

$$\mathfrak{F}(\mathfrak{x}) = \int_D K(s, t)x(t)\, dt,$$

where s, t denote points of the bounded domain D in a finite dimensional Euclidean space, or of the form $\mathfrak{x} + \mathfrak{F}(\mathfrak{x})$. It is also well known (see [4; Chapter III]) that in the case of a symmetric kernel $K(s, t)$ the theory of these operators is closely connected with the existence of extrema of the integral form

$$I(\mathfrak{x}) = \tfrac{1}{2} \iint\limits_{DD} K(s, t)x(s)x(t)\, ds\, dt \qquad [i(\mathfrak{x}) = \tfrac{1}{2}(\mathfrak{x}, \mathfrak{x}) + I(\mathfrak{x})],$$

where $(\mathfrak{x}, \mathfrak{y}) = (x(t), y(t))$ denotes the scalar product. Since the scalar product $(\mathfrak{F}(x(t)), h(t))$ is the Fréchet differential $D(\mathfrak{x}, \mathfrak{h})$ of the integral form $I(\mathfrak{x})$ these connections have been generalized to completely continuous but not necessarily linear operators $\mathfrak{F}(\mathfrak{x})$ in essentially the following way: such an $\mathfrak{F}$ considered as an operator in a Hilbert space is called symmetric (see [10], [14]) if there exists a scalar (*i.e.*, a real-valued function) $I(\mathfrak{x})$ in H such that the scalar product $(\mathfrak{F}(\mathfrak{x}), \mathfrak{h})$ coincides with the Fréchet differential $D(\mathfrak{x}, \mathfrak{h})$ of $I(\mathfrak{x})$. $\mathfrak{F}$ is called the gradient of I (see [7; 67]), and the mapping $\mathfrak{y} = \mathfrak{F}(\mathfrak{x})$ of H (or part of H) into H is called a gradient mapping. The applicability of these concepts to problems of analysis is due to the fact that in a critical point of the scalar $I(\mathfrak{x})$ $[i(\mathfrak{x})]$ for some suitable λ,

$$\mathfrak{F}(\mathfrak{x}) = \operatorname{grad} I(\mathfrak{x}) = \lambda\mathfrak{x} \qquad [\mathfrak{x} + \mathfrak{F}(\mathfrak{x}) = \operatorname{grad} i(\mathfrak{x}) = \lambda\mathfrak{x}],$$

where $\lambda = 0$ if $\mathfrak{x}$ is an interior point of the domain in the Hilbert space considered; and various authors have proved existence theorems concerning equations of the form $\mathfrak{F}(\mathfrak{x}) = \lambda\mathfrak{x}$ $[\mathfrak{x} + \mathfrak{F}(\mathfrak{x}) = \mathfrak{o}]$ by proving the existence of an extremum for the scalar $I(\mathfrak{x})$ $[i(\mathfrak{x})]$. (See [7], [10], [13], [14], [15].)

The present paper deals with scalars and gradient mappings in Banach spaces; part of it may be considered as an extension to such spaces of the main results of [12]. (The feasibility of such an extension was suggested by T. H. Hildebrandt.) Since in a Banach space E the Fréchet differential $D(\mathfrak{x}, \mathfrak{h})$ of I

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