THE REAL ELLIPTIC ϑ_3 -FUNCTION

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1. For real x and for

$$0 < q < 1, \tag{1}$$

let $\theta_q(x)$ denote that theta-function which is ϑ_3 in Jacobi's notations but is made to have the period 2π (instead of the customary period, which is 1). Thus

$$\theta_q(x) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos nx. \tag{2}$$

If $q = e^{-t}$ and $x = \phi$, where t is a time variable and ϕ an angular coordinate, then (2) represents the "Green" or "source" function of Fourier's boundary value problem for the cooling of a ring-shaped wire. (See [3; Chapter V] or [1; 262–264 and footnote on 259–260].) Considerations based on this interpretation, or on the corresponding statistical restatement [5; 477–478] of (2), suggest the following assertion: *For every q-value contained in the range* (1), *the graph* $y = \theta_q(x)$ *of the* (*even, periodic*) *function* (2) *has just one pair of flex points over a period*; more specifically, d^2y/dx^2 is negative or positive according as $0 \leq |x| < x_q$ or $x_q < |x| \leq \pi$, where x_q is a unique point of the interval $0 < x < \pi$ whenever $0 < q < 1$.

If q is very close to the lower end of the region (1), such a division of the graph into convex and concave arches is made almost evident by the preponderance of the first term, $q \cos t$, of the series (2). When q increases, the convergence of (2) deteriorates and, if q is close to the upper end of (1), the series is conveniently replaced by that expansion of $\theta_q(x)$ which is supplied by the linear transformation formula of the ϑ_3-function, that is, by the identity

$$\theta_q(x) = 2\pi^{\frac{1}{2}}/p \sum_{n=-\infty}^{\infty} \exp[-(x + 2\pi n)^2/p^2], \tag{3}$$

where

$$p = (-4 \log q)^{\frac{1}{2}} \qquad (0 < p < \infty;\ 1 > q > 0). \tag{4}$$

But all of this, along with the corresponding numerical graphs based on tables [2; 44], fails to prove the truth of the italicized assertion, a proof of which is the object of this note.

2. For a fixed q, differentiate (3) twice with respect to x. Since (4) does not contain x, this gives

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