DEFINITE INTEGRAL SYSTEMS

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1. **Introduction.** Wilkins [8] has recently given a definition of definite self-conjugate adjointness for a system of integral equations written in matrix form as

$$y(x) = \lambda \int_a^b K(x, t)y(t)\, dt, \tag{1.1}$$

where $K(x, t) \equiv H(x, t)S(t)$. This was a weakening of a definition as first formulated by Reid [4] for systems (1.1) in which the elements of $K(x, t)$ are real-valued.

For each of the integral systems treated in [4] and [8] the definiteness property of the system was imposed on the matrix $S(x)$. Now if $y(x)$ is a solution of (1.1) corresponding to a value λ then

$$J[y] \equiv \int_a^b \int_a^b y^*(x)S(x)K(x, t)y(t)\, dx\, dt = (1/\lambda) \int_a^b y^*(x)S(x)y(x)\, dx, \tag{1.2}$$

where $y^*(x)$ denotes the conjugate transpose of $y(x)$. As it readily follows that the definiteness property of $S(x)$ could equally well have been phrased as a definiteness property of $\int_a^b y^*Sy\, dx$, relation (1.2) suggests considering systems (1.1) for which the definiteness property is placed on $J[y]$. Moreover, this study yields further results for the definite systems of [8]. Furthermore, these systems include the class of integral systems to which an H-definitely self-conjugate adjoint differential system of Reid [6] is equivalent. For the analogous situation in definite boundary value problems see [6].

In §§2—5 the above notions of definiteness are extended to integral systems (1.1) in which no restriction is made on the form of the kernel $K(x, t)$. Preliminary results are presented in §2, and the definitions of the two types of definite integral systems, termed definitely self-conjugate adjoint and J-definite, are given in §3. Some fundamental properties of definitely self-conjugate adjoint and J-definite integral systems, such as the reality of the characteristic values, the equality of their index and multiplicity, and a type of completeness property of the totality of the characteristic solutions, are contained in §4. In §5 additional results are obtained for a special definitely self-conjugate adjoint integral system treated in [8; §7]. §6 and §7 are devoted to a consideration of definite integral systems (1.1) whose kernel matrix $K(x, t)$ is of the form $H(x, t)S(t)$. In §6 further results for such definitely self-conjugate adjoint integral systems are obtained, while in §7 extremizing properties of the charac-

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