RINGS WITH ADDITIVE GROUP WHICH IS THE DIRECT SUM OF CYCLIC GROUPS

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1. **Introduction.** The purpose of this note is to construct all rings which have as their additive group, a given additive abelian group which is the direct sum of cyclic groups. The method is an easy generalization of the usual method of determining the algebras over a vector space. (See [1].)

If an abelian group G contains a set of subgroups $[H_i]$ such that (i) $H_t \cap \{[H_i]\} = 0$ for $i \neq t$, and (ii) any element g in G can be written as a finite sum, $g = h_{i_1} + h_{i_2} + \cdots + h_{i_k}$, where h_{i_j} is in H_{i_j}, then G is the direct sum of the groups H_i. In particular, if the H_i are cyclic groups, their generators u_i are a basis for G over the domain of integers which is an operator domain for every additive abelian group.

Infinite cyclic groups will be said to have order zero, and congruences modulo zero are ordinary equalities. We will define the symbol $(a_1, a_2, \cdots, a_k)$ to be zero if $a_i = 0$, $i = 1, \cdots, k$, and to be the greatest common divisor of the non-zero a_i if not all of the a_i are zero.

2. **The construction of rings with given additive group.** Let G be an additive abelian group which is the direct sum of cyclic groups $\{u_i\}$ of order t_i. Any two elements g, f in G have the following representations which are not necessarily unique:

$$g = \sum_i a_i u_i , \qquad f = \sum_j b_j u_j ,$$

where the a_i and b_j are integers such that almost every $a_i \equiv 0 \pmod{t_i}$ and almost every $b_j \equiv 0 \pmod{t_j}$.

Theorem. *G is made into a ring if, and only if, for given integers g_{ijk}, multiplication is defined by the formula*

$$(1) \qquad gf = (\sum_i a_i u_i)(\sum_j b_j u_j) = \sum_{ijk} g_{ijk} a_i b_j u_k$$

such that,

(a) *for a given i and j almost every $g_{ijk} \equiv 0 \pmod{t_k}$;*
(b) $\sum_k g_{ijk} g_{kmp} \equiv \sum_k g_{jmk} g_{ikp} \pmod{t_p}$;
(c) $g_{ijk} \equiv 0 \pmod{t_k/(t_i, t_j, t_k)}$, *where (t_i, t_j, t_k) is the greatest common divisor of the non-zero t's in the symbol and $0/(0, 0, 0) = 1$.*

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