PRESERVATION OF DIVISIBILITY IN QUOTIENT GROUPS

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1. **Introduction.** The main purpose of this paper is to show that certain additive abelian groups can never be compact topological groups. This is so because the closed subgroups of an abelian compact group have a certain special algebraic property expressible in terms of divisibility. Closely related to this is the fact that a torsion-free abelian compact group has a non-zero, at least everywhere dense subgroup which is "like" the subgroup of torsion elements of an abelian group, the defining condition for this subgroup again being stated in terms of divisibility. Direct sums of copies of the integers, for instance, do not have these special subgroups and, consequently, cannot admit topologies which make them compact topological groups. This conclusion is well known for the countable case.

We shall consider only additive abelian groups. Let g be an element of such a group G, and let n be a non-zero integer. Then we say that n divides g, $n \mid g$, if there exists a solution in the group G to the equation $nx = g$. Let p be a prime, and let n be a non-negative integer. If $p^n x = g$ has a solution $x = g_0 \varepsilon G$, and if $p^{n+1}x = g$ has no solution in G, we say, following Prüfer [8], that g has the height n at p in the group G. We write $\Gamma_p(g; G) = n$, or, if there be no ambiguity, $\Gamma_p(g) = n$. If there exist elements of G, x_n, $n = 0, 1, 2, \cdots$, such that $p^n x_n = g$, then we say that the height of g at p is infinite in G and write $\Gamma_p(g; G) = \infty$ (or $\Gamma_p(g) = \infty$). For an integer n, we usually write $\Gamma_p(n)$ rather than $\Gamma_p(n; I)$, where I is the additive group of integers.

With the convention that $\infty > n$ for any integer n, it follows immediately from the definitions that

(1) $\Gamma_p(x; G) > \Gamma_p(y; G)$ implies $\Gamma_p(x + y; G) = \Gamma_p(y; G)$;

(2) $\Gamma_p(x; G) = \Gamma_p(y; G)$ implies $\Gamma_p(x + y; G) \geq \Gamma_p(y; G)$;

(3) $\Gamma_p(n) = 0$ implies $\Gamma_p(ng; G) = \Gamma_p(g; G)$, where n is an integer and $g \varepsilon G$;

(4) $\Gamma_p(n) > 0$ and $\Gamma_p(g; G) < \infty$ imply $\Gamma_p(ng; G) > \Gamma_p(g; G)$; and $\Gamma_p(g; G) = \infty$ implies $\Gamma_p(ng; G) = \infty$ for all integers n.

If H is a subgroup of a group G, we define the division hull (see [1; 554 ff.]) H^0 of H in G to be the set of all elements of G such that $h \varepsilon H^0$ implies that there exists a non-zero integer n with $nh \varepsilon H$. It is clear that H^0 is a subgroup of G. If $H^0 = H$, we say that the subgroup H is a subgroup with division [1; 554 ff.] in G. It is easy to show, for instance, that the torsion elements of a group form

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