# INTEGRAL REPRESENTATIONS OF REMAINDERS

By Arthur Sard

1. **Introduction.** Many of the processes of computation are linear on a space of functions and exact for a class of functions on which a certain linear operation vanishes. Examples of such processes are the following: the approximation of a definite integral by a linear combination of values of the integrand and its derivatives, the approximating formula being such as to be exact if the integrand is a polynomial of degree $n - 1$; the approximation of a function by that polynomial of degree $n - 1$ which minimizes a weighted integral of the square of the error; the approximation of a function by a Lagrangian interpolating polynomial of degree $n - 1$ or by the first $n$ terms of its Taylor series; and so on.

In each of the cases cited, the approximating process is exact for functions whose derivatives of order $n$ vanish. In 1913 Peano [7], [8] pointed out that the error committed can then be expressed as a single integral of the $n$-th derivative. A precise statement of this fact is given in Theorems 2 and 4 below.

In 1939 Rémès [10], [11] generalized Peano's theorem to the case in which the approximating process is exact for functions satisfying a linear homogeneous differential equation. Peano's theorem is the special case of Rémès' theorem obtained when the linear homogeneous differential equation on the function $x(s)$ is the equation $x^{(n)}(s) = 0$.

Theorem 1 of the present paper offers the further generalization to the case in which the approximating process is a linear operation which is exact for all functions on which another linear operation vanishes. Theorem 1 affords what may be considered a broad and useful characterization of the situation in which integral remainders are accessible. Known theorems on the representation of linear functionals provide algorithms (in the sense of explicit direct procedures) for the integral representations. Rémès' theorem is the special case of Theorem 1 obtained when the operation $V$ is a linear functional on the space $C_n$ and the operation $U$ is that of forming a linear homogeneous differential expression of order $n$ with leading coefficient unity.

Applications are given in §3.

The general point of view adopted in this paper may be considered useful in that it establishes similarities in mathematical situations that might otherwise appear unrelated. The similarities are often suggestive. It should not be expected that the general theory will afford results which cannot be obtained directly. A direct attack on any problem is at least as powerful as any other attack, inasmuch as the direct attack can include a particularization of any general argument.

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