THE COMPLETE MONOTONICITY OF CERTAIN FUNCTIONS DERIVED FROM COMPLETELY MONOTONE FUNCTIONS

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1. Introduction. We prove the following theorems for any function $F(x)$ which is completely monotonic and has derivatives for $0 \leq x \leq \infty$. That is, we assume throughout that

$$(-1)^k F^{(k)}(x) \geq 0 \qquad (0 \leq x \leq \infty). \tag{1}$$

THEOREM A. *If we define*

$$F_{m,n}(x) = \begin{vmatrix} F^{(m)}(0) & F^{(m+1)}(0) & \cdots & F^{(m+n)}(0) \\ F^{(m+1)}(0) & F^{(m+2)}(0) & \cdots & F^{(m+n+1)}(0) \\ \vdots & \vdots & & \vdots \\ F^{(m+n-1)}(0) & F^{(m+n)}(0) & \cdots & F^{(m+2n-1)}(0) \\ F^{(m)}(x) & F^{(m+1)}(x) & \cdots & F^{(m+n)}(x) \end{vmatrix}, \tag{2}$$

then

$$\frac{(-1)^m}{x^n} F_{m,n}(x) \tag{3}$$

is completely monotonic for $0 \leq x \leq \infty$.

THEOREM B. *If we choose constants* λ_i *and* c_i *with the* $c_i \geq 0$ *so that*

$$F(x) - \sum_{i=1}^{n} \lambda_i e^{-c_i x} \tag{4}$$

and its first $2n - 1$ *derivatives all vanish at the origin, then*

$$\frac{1}{x^{2n}} \left\{ F(x) - \sum_{i=1}^{n} \lambda_i e^{-c_i x} \right\} \tag{5}$$

is completely monotonic for $0 \leq x$, *and* $\lambda_i \geq 0$ *for* $1 \leq i \leq n$.

THEOREM C. *If we choose constants* λ_i *and* c_i *with the* $c_i \geq 0$ *so that*

$$F(x) - \lambda_0 - \sum_{i=1}^{n} \lambda_i e^{-c_i x} \tag{6}$$

Received November 6, 1947.