A NEW GEOMETRICAL INTERPRETATION OF THE LEBESGUE AREA OF A SURFACE

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1. **Introduction.** The purpose of this paper is to give a new geometrical interpretation of the Lebesgue area of a Fréchet surface of the type of the circular disc. The reader is referred to Youngs [7] for the definition of such a surface. The motivation for this new interpretation arises from work done by Deltheil [1] and Favard [2] on a two-dimensional measure for sets in Euclidean 3-space E_3 and on the application of this measure to what may be called the case of elementary surfaces.

The basis of the work of Deltheil and Favard is an invariant measure in the space of the lines in E_3. However, it will be more convenient for us to use *directed lines.* The set G of the directed lines in E_3 can be so metrized that each distance preserving transformation of E_3 induces a topological transformation in G. From general measure theory, there exists in G a unique (up to a constant multiplying factor) measure μ which is invariant under these induced topological transformations. Since we shall need an explicit formula for μ, we develop such a formula in §3.

For a set E in E_3, let $N(g)$ denote the number of intersections of a directed line g with E. If $N(g)$ is a summable function on G with respect to the invariant measure μ, then, for μ properly normalized, the Favard measure $F(E)$ of E is given by the formula

$$F(E) = \int_G N(g)\, d\mu.$$

If a point set E is a surface in the sense used in classical differential geometry then it follows readily that the area in the classical sense agrees with the Favard measure.

In the remainder of this paper the term *surface* will always refer to a *Fréchet surface of the type of the circular disc.* For each directed line g and surface S we define (see §6) a function $N_e(g, S)$, to be termed the essential number of intersections of the directed line g with the surface S, in terms of ϵ-deformations of S. We then prove (see §7) that, for the properly normalized invariant measure μ in the space G of the directed lines in E_3, the Lebesgue area $A(S)$ of a surface S is given by the formula

$$A(S) = \begin{cases} \int_G N_e(g, S)\, d\mu \text{ if } N_e(g, S) \text{ is summable,} \\ +\infty \text{ otherwise.} \end{cases}$$

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