A RENEWAL THEOREM

By David Blackwell

1. **Introduction.** Let x_i be independent non-negative chance variables with identical distributions. The asymptotic behavior of the expected number $U(T)$ of sums $s_k = x_1 + \cdots + x_k$ lying in the interval $(0, T)$ has been studied by Feller [2], using the integral equation of renewal theory and the method of Laplace transforms. Recently Doob [1] has obtained as a consequence of general theorems on stationary Markov processes the following result: if the distribution of some s_k is non-singular, then $U(T + h) - U(T) \to h/E(x_1)$ as $T \to \infty$ for every $h > 0$. Täcklind [4] has obtained an excellent estimate for $U(T)$ itself: when the k-th moment of x_1 exists for some $k > 2$ and the values of x_1 are not all integral multiples of some fixed constant, his estimate shows at once that $U(T + h) - U(T) \to h/E(x_1)$.

In this paper we shall prove the following

Theorem. *Unless all values of x_1 are integral multiples of some fixed constant,*

$$U(T + h) - U(T) \to h/E(x_1) \qquad (T \to \infty)$$

for every $h > 0$. (If $E(x_1) = \infty$, then $h/E(x_1)$ is to be interpreted as zero.)

The case excluded in our theorem is essentially that of integral-valued chance variables; here a corresponding result (with minor complications due to periodicity) has been obtained by Feller (oral communication), using a general theorem on power series due to Erdös, Feller, and Pollard. Thus our result complements that of Feller: together they describe the limits of $U(T + h) - U(T)$ in every case. Our principal tool (Theorem 1) is obtained by the method of Erdös, Feller, and Pollard; it is in a sense weaker than a result of Doob [1], but suffices to prove our theorem (which implies both results, in the case here considered).

2. **Definitions and preliminaries.** For any chance variable z and any $h > 0$ we define $N_k(z, h)$ as the number of sums $x_{k+1}, x_{k+1} + x_{k+2}, \cdots$ lying in the interval $z \leq s < z + h$, and define $N(a, h) = N_0(a, h)$. For any constants a, h the chance variables $N_k(a, h)$ have distributions independent of k; we define $U(a, h) = E[N_k(a, h)]$. Thus $U(0, T)$ is the function $U(T)$ defined in the introduction: the expected number of sums $s_k = x_1 + \cdots + x_k$ for which $0 \leq s_k < T$. We shall sometimes write $U(z, h)$ where z is a chance variable; $U(z, h)$ is then itself a chance variable, assuming the value $U(a, h)$ when $z = a$.

Lemma 1. *$U(a, h)$ is finite for all a, h.*

This follows immediately from a result of Stein [3], who has shown that $P\{s_k < b\} \to 0$ exponentially as $k \to \infty$ for every constant b.

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