NOTE ON PALEY-WIENER'S THEOREM

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1. Paley and Wiener proved the following theorem [1; 75].

Let $f(z)$ be an even entire function of order not exceeding 1 *and let the number of its roots* $\pm z_n$ *within a circle radius* r *about the origin be* $2\lambda(r)$. *Let*

$$\lambda(r) \sim Br; \tag{1}$$

then all the roots of $f(z)$ will be real, when and only when

$$B = -\frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\log |f(x)|}{x^2}\, dx.$$

The object of this note is to prove the following extension: if we replace the condition (1) of Paley-Wiener's result by

$$\lambda(r) = Ar\log r + Br + o(r) \tag{2}$$

as $r \to \infty$, then all roots of $f(z)$ will be real, when and only when

$$-\frac{2}{\pi^2}\int_0^T \frac{\log |f(x)|}{x^2}\, dx = A\log T + B + o(1)$$

as $T \to \infty$.

In order to prove this result, let us write

$$\phi(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right)$$

where $\lambda_n = |z_n|$. By Paley-Wiener's method we have only to show that

$$-\frac{2}{\pi^2}\int_0^T \frac{\log |\phi(x)|}{x^2}\, dx = A\log T + B + o(1),$$

as $T \to \infty$.

Put

$$N(t) = \frac{1}{t}\int_0^{1/t} x^{-2}\log|1 - x^2|\, dx = -\log|1 - t^{-2}| - t^{-1}\log\left|\frac{1+t}{1-t}\right|;$$

then we have

$$N'(t) = t^{-2}\log\left|\frac{1+t}{1-t}\right|,$$

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