THE SINGULAR ELEMENTS OF A BANACH ALGEBRA

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1. **Introduction.** A *Banach algebra*, or simply *B-algebra*, is a linear algebra (not necessarily commutative) in which the underlying vector space is a complex Banach space. The norm of the Banach space is related to multiplication by the condition $\| xy \| \leq \| x \| \cdot \| y \|$. The existence of an identity element e, with $\| e \| = 1$, will also be assumed. Banach algebras are also called normed rings [4].

We outline first a few properties of a commutative B-algebra $\mathfrak{A}$ which were obtained by I. Gelfand [4]. Let $\mathfrak{M}$ denote the class of all non-trivial maximal ideals in $\mathfrak{A}$. Then each $M \in \mathfrak{M}$ determines a homomorphism $x(M)$ of $\mathfrak{A}$ onto the complex numbers. The kernel of the homomorphism is M; that is, M consists of all x such that $x(M) = 0$. The class $\mathfrak{M}$ can be topologized so that it becomes a bicompact topological space on which $x(M)$ is a continuous function of M for each $x \in \mathfrak{A}$. Also, $\max | x(M) | = \lim \| x^n \|^{1/n}$, for every x. In general, $\max | x(M) | \leq \| x \|$. However, if $\| x^2 \| = \| x \|^2$ in $\mathfrak{A}$, then $\max | x(M) | = \| x \|$ and convergence in $\mathfrak{A}$ is equivalent to uniform convergence of the associated functions on $\mathfrak{M}$.

Two B-algebras are said to be *equivalent* provided there exists a one-to-one correspondence between them which preserves the norm as well as the algebraic operations. A subalgebra $\mathfrak{A}'$ of a B-algebra $\mathfrak{A}$, with the same identity element as $\mathfrak{A}$, is called a *B-subalgebra* of $\mathfrak{A}$ provided it is topologically closed in $\mathfrak{A}$. On the other hand, a B-algebra $\mathfrak{A}'$ is called a *B-extension* of $\mathfrak{A}$ provided $\mathfrak{A}$ is equivalent to a B-subalgebra of $\mathfrak{A}'$. In this case $\mathfrak{A}$ is also said to be *embedded in* $\mathfrak{A}'$.

An element of a B-algebra $\mathfrak{A}$ is said to be *left* (*right*) *regular* provided it possesses a left (right) inverse in $\mathfrak{A}$. If x is both left and right regular, then it possesses a unique inverse and is said to be *regular*. The class of all (left, right) regular elements is denoted by (G^l, G^r) G. The class G is a group under multiplication and is an open subset of $\mathfrak{A}$. In particular, if $\| e - x \| < 1$, then $x \in G$ [4]. The component G_e of the open set G, which contains the identity element e, is called the *principal component* of G and is also a group [9]. An element which is not (left, right) regular is said to be (*left, right*) *singular*. The class of all (left, right) singular elements is denoted by $(S^l, S^r)S$. If S consists of only the zero element, then $\mathfrak{A}$ reduces to the complex numbers [4]. If x is singular in every B-extension of $\mathfrak{A}$, then x is said to be *permanently singular*.

The set $\sigma(x)$ of all complex numbers λ such that $x - \lambda e$ is singular is called the spectrum of x. The spectrum of x is a bounded, closed subset of the complex plane; in fact, if $\lambda \in \sigma(x)$, then $| \lambda | \leq \| x \|$. If $x - \lambda e$ is permanently singular,

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