COHOMOLOGY AND REPRESENTATIONS OF ASSOCIATIVE ALGEBRAS

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Introduction. In this paper we continue our study of the cohomology theory for associative algebras. (There are two previous papers by the author on this subject: [2] and [3].) We begin by showing quite explicitly that the cohomology theory simply amounts to a reformulation of a part of representation theory; more precisely, that it is equivalent to the study of the decomposability of two-sided modules (§1; also, see Theorem 2.3).

There is quite a variety of problems in linear algebra whose analysis depends essentially on the decomposability of certain reducible modules, although the problems themselves do not contain any reference to these modules. In such cases, the cohomology theory is a suitable tool because it is frequently quite easy to extract the critical cocycle (or "obstruction") from the given situation, while the critical module may be rather remote. Generally speaking, the introduction of the cohomology groups greatly increases the flexibility of representation theory.

Section 1 is devoted to an interpretation of the cohomology groups of arbitrary dimension as groups of "module enlargements". This development yields some useful formal tools which are applied repeatedly in the sequel. In §2 we discuss a few simple applications, chiefly for illustrative purposes.

Apart from the general interpretation of §1, there are entirely different interpretations for the cases of dimension 2 and 3 which are of special interest. In this connection, the real innovation is an interpretation, due to Eilenberg and MacLane, of the three-dimensional cohomology groups (see [1]).

Actually, these authors have given such an interpretation for group theory. The results for algebras, and their proofs, are quite analogous. However, since here we have to deal with a fuller structure, the situation involves some features which are absent in the case of groups. It is therefore desirable to introduce an independent set of notions appropriate to the theory of linear algebras. These are given in §§3 and 4. The resulting theory may be regarded as a generalization of the ordinary representation theory.

In §§5, 6, and 7 we adapt the results of Eilenberg and MacLane to linear algebra. Since these results are tied up with the theory of extensions and the two-dimensional cohomology groups, our treatment includes an interpretaton of the two-dimensional cohomology groups as groups of translations operating on certain sets of mutually related extensions. The proof that the Eilenberg-MacLane interpretation is complete presents some special difficulties in the case of algebras. It necessitates a rather elaborate construction which is given in §8.

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