A MINIMAL PROBLEM FOR HARMONIC FUNCTIONS

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1. Let n be a positive integer. Let $h_1, h_2, \cdots, h_n$ be arbitrary real numbers; and let $\theta_1, \theta_2, \cdots, \theta_n$ be arbitrary real numbers such that

$$0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi.$$

Put $\theta_{n+1} = \theta_1$, $h_{n+1} = h_1$; and let $\lambda_1(\theta)$, $\lambda_2(\theta)$ be the step functions of period 2π determined by

$$\lambda_1(\theta) = \min (h_k, h_{k+1}), \qquad \lambda_2(\theta) = \max (h_k, h_{k+1}) \qquad (\theta_k \leq \theta < \theta_{k+1})$$

for $k = 1, 2, \cdots, n$. Let S: $r = 0$, $\theta = 0$ $(0 < r < 1, 0 \leq \theta < 2\pi)$ denote the interior of the unit circle $r = 1$. Let Γ denote the class of functions, $U(r, \theta)$, each of which is harmonic in S, and satisfies

$$\lambda_1(\theta) \leq \underline{\lim}_{r \to 1-} U(r, \theta), \qquad \overline{\lim}_{r \to 1-} U(r, \theta) \leq \lambda_2(\theta)$$

for almost all θ, $0 \leq \theta < 2\pi$. We consider the problem of minimizing in Γ the functional

$$A[U] = \iint_S |\nabla U|^2 \, dS,$$

where $|\nabla U|$ is the magnitude of the gradient of U. Our analysis is based on Douglas' methods in his paper [1].

Our chief results for the general case, $1 \leq n$, can be summarized as follows. In Theorem I we prove that the greatest lower bound, A_0, of $A[U]$ for U in Γ is finite. In Theorem II we prove that $A[U]$ assumes the value A_0 at least once in Γ. In Theorem III we prove that $A[U]$ assumes the value A_0 at most once in Γ, except when n is even and all the h_k of even index are greater than, or are less than, all those of odd index. In Theorem IV we obtain a representation for U in Γ, $A[U] = A_0$, of $[zF'(z)]^2$, where $z = re^{i\theta}$ and F has U as real part. We conclude the paper with a discussion of the case $n = 4$, which is the first case of interest. If $n = 1, 2, 3$, then $A_0 = 0$. For $n = 1$ this value is assumed in Γ on and only on $U \equiv h_1$; for $n = 2$, on and only on $U \equiv h$, h const., $\min (h_1, h_2) \leq h \leq \max (h_1, h_2)$; for $n = 3$, on and only on $U = h''$, where h'' is the middle one of the three numbers h_1, h_2, h_3 when the latter are arranged in ascending order, $h' \leq h'' \leq h'''$.

2. We note first the following

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