DUALITY IN LINEAR SPACES

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1. **Introduction.** Given any convex topological linear space X, we let X^* denote the class of all those linear functionals defined on X which are continuous with respect to the topology of X. We refer to X^* as the *adjoint* of X. One may select a topology t which converts X^* into a convex topological linear space X^{*t}. The convexity in X insures that non-zero elements of X represent non-vanishing functionals on X^* according to the formula

$$\xi(f) = f(x) \qquad (x \in X, f \in X^*).$$

A first question is whether the elements of X represent exactly the entire adjoint X^{*t*} of X^{*t}. Let t be called *reflexive* if this is true; then our answer may be stated as follows. Let p be the *weak* topology for X^*; and let κ be that topology for X^* according to which convergence of functionals means uniform convergence on the convex, weakly compact subsets of X. Then t is reflexive if and only if its family of open sets includes that of p and is included in that of κ (Theorem 2).

Supposing t to be reflexive, we next consider topologizing X^{*t*}. One might ask for a simple condition on a topology u insuring that X^{*t*u} is *topologically* isomorphic with X. This formulation is too vague, because it admits the following trivial answer: let us simply adopt in X^{*t*} the topology of X. An objection to this *ad hoc* solution lies in the fact that in topologizing the adjoint, X^{*t*}, of X^{*t}, we have not confined our attention to these two spaces alone. A rule T which associates with each topological linear space a topology *for its adjoint*, and converting this adjoint into a topological linear space, will be called, somewhat briefly, an *assignment of topologies.* To illustrate the term and its notation: when T is applied to X we obtain the T-topologized adjoint $X^{*T(X)}$; and when applied to X^{*t}, we obtain $X^{*t*T(X^{*t})}$.

Such an assignment T will be called *covariant* if, whenever there exists a *continuous* linear operation φ from X to another topological linear space Y, then the dual mapping φ^*, from $Y^{*T(Y)}$ to $X^{*T(X)}$, defined by

$$\varphi^*(f)(x) = f(\varphi(x)) \qquad (x \in X, f \in X^*),$$

is also continuous. We remark that the p- and κ-topologies are covariant.

The problem next to be considered can now be formulated. Does there exist any covariant assignment of topologies T such that for any two members X and Y of the class $\mathfrak{L}$ of all topological linear spaces, $Y \cong X^{*T(X)}$ if and only if $X \cong Y^{*T(Y)}$?

We have not been able to settle this question for the entire class $\mathfrak{L}$; but for

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