A SPECIAL CLASS OF FUNCTIONS WITH POSITIVE REAL PART IN A HALF-PLANE

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I. **Introduction.** This paper contains an investigation of the properties of a class of functions, which has recently arisen in connection with the theory of ultra-high-frequency electro-magnetic impedances. The class consists of those complex functions $Z(s) = R(s) + iX(s)$ of a complex variable $s = \gamma + i\omega$ which satisfy the following conditions. (An asterisk denotes complex conjugation.)

(1)
- (a) $Z(s)$ is analytic and single-valued in $\gamma > 0$.
- (b) $Z(s)$ has no singularities other than poles on $\gamma = 0$.
- (c) $Z(s^*) = Z^*(s)$ (*i.e.* real on the real axis).
- (d) $R \geq 0$ in $\gamma \geq 0$.

We shall, for brevity, write $Z = PRF$ to mean that Z satisfies (1). For simplicity in the statements of the theorems, we shall also assume that Z is not identically zero. The letters "*PRF*" stand for "positive real function", a term due to Brune [1]. The name is intended to be suggestive rather than precise; "positive" refers to (1d), "real", to (1c). We shall also use the abbreviation $Z = iPRF$ for $Z = PRF$ and $R \equiv 0$ on $\gamma = 0$.

Functions satisfying (1a, d) have been extensively studied in mathematical literature. In physics literature, the restriction that Z be a rational function is usually added (lumped-constants circuits). The present class lies between these two and possesses many properties analogous to those of the rational functions.

The purpose of the present paper is both to provide an exposition of results which have not been given a unified treatment before and also to present some theorems which are believed to be new. These latter arise almost entirely from the use of condition (1b) and include Theorems 1, 6 through 12 inclusive, and their corollaries. Of these, Theorems 1, 6, 7, 8 follow upon fairly evident application of standard results.

II. **Basic theorems.** First note that (1d), when combined with the fundamental property of harmonic functions, implies that Z has no zeros in $\gamma > 0$. Accordingly, $1/Z = PRF$ if and only if $Z = PRF$.

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