THE BOUNDEDNESS OF SOLUTIONS OF INFINITE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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1. **Introduction.** In this paper we wish to discuss the existence, boundedness, and uniqueness of solutions of infinite-order systems of linear differential equations. These systems have the form

$$\frac{dx_i}{dt} = \sum_{j=1}^{\infty} a_{ij}x_j \qquad (i = 1, \cdots, \infty). \tag{1.1}$$

The x_i are real functions of the independent variable t, which is taken over the interval $(0, \infty)$.

At first glance, the condition of linearity might seem unduly restrictive, but as we shall show below, linear equations of this form include all finite-order non-linear systems of analytic type, and it is easy to show that the apparently more general system

$$\frac{dy_i}{dt} = \sum_{j=1}^{\infty} a_{ij}y_j + \sum_{j,k=1}^{\infty} a_{ijk}y_jy_k + \cdots, \tag{1.2}$$

can also be written as an equation of the form (1.1).

It is convenient to introduce vector-matrix notation, and to consider, instead of (1.1), the equation

$$\frac{dx}{dt} = Ax \qquad (x(0) = x_0), \tag{1.3}$$

where A is the infinite matrix, (a_{ij}), $i, j = 1, \cdots, \infty$, and x is the infinite column vector whose components are the x_i, $i = 1, \cdots, \infty$.

In a recent paper, Arley and Borchsenius [1], investigated the solutions of (1.3), and showed that under certain conditions on the matrix A and the initial vector, x_0, a solution of (1.3) could be written

$$x = x_0e^{At} \tag{1.4}$$

taking A to be a constant matrix.

This form, (1.4), is the analogue of the form of the solution of a finite-order linear system with constant matrix.

Although this representation is very elegant, it seems to suffer from three defects. In the first place, a strong restriction is placed upon A, namely that $\sum_{i,j=1}^{\infty} | a_{ij} | < \infty$; secondly, as far as calculations are concerned, it furnishes no constructive means of finding the solution; thirdly, it offers no clue as to

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