THE INTEGRAL TRANSFORMS WITH ITERATED LAPLACE KERNELS

By Harry Pollard

1. **Introduction.** The successive iterates of the Laplace kernel $G_0(x, y) = e^{-xy}$ are defined by the recursion formula

$$(1.1) \qquad G_n(x, y) = \int_0^\infty e^{-xt} G_{n-1}(t, y)\, dy \qquad (n = 1, 2, \cdots).$$

These iterates fall into two classes, depending on the parity of n: the kernels $H_n(x, y) = G_{2n-1}(x, y)$ are homogeneous of degree -1, while the kernels $K_n(x, y) = G_{2n}(x, y)$ are functions of xy.

Recently the author [3] has obtained a solution of the inversion problem for the transforms

$$f(x) = \int_{0+}^\infty H_n(x, y)\, d\alpha(y),$$

where $\alpha(y)$ is of bounded variation in every finite interval on the positive axis. In the present paper we shall treat the analogous problem for the transforms

$$(1.2) \qquad f(x) = \int_{0+}^\infty K_n(x, y)\, d\alpha(y).$$

The case $n = 0$ proves to be exceptional in some ways, so that we restrict ourselves to $n = 1, 2, \cdots$; in any event, for $n = 0$ the transform reduces to the classical Laplace integral, for which the inversion theory is well known [6]. If $\alpha(y)$ is of the form $\int_0^y \varphi(u)\, du$, where $\varphi(u) \in L^2(0, \infty)$, then an inversion formula for the transforms (1.2) is known [2]; our present methods require no restriction on $\alpha(y)$ beyond the convergence of the integral (1.2).

It is first necessary to obtain a certain amount of information concerning the behavior of the kernels $K_n(x, y)$. Widder [5] has studied this problem in the real domain, but our methods call for information in the complex domain also. This is a more awkward problem than the analogous one for the kernels $H_n(x, y)$. In the latter case simple explicit formulas for the kernels exist in terms of the elementary functions or of the gamma function [3], [5]. For the kernels $K_n(x, y)$ nothing so simple is available, and we must be content with asymptotic approximations.

Since $K_n(x, y)$ is a function of xy we may write it in the form $K_n(xy)$, where $K_n(x) = K_n(x, 1)$. The function $K_n(z)$, $z = x + iy$, turns out to be an entire function of $\log z$. This makes the entire function $h_n(z) = K_n(e^z)$ the more natural to study directly. This is done in §2, where for the sake of completeness we prove somewhat more than is necessary for the present paper.

Received April 14, 1947.