TWO-DIMENSIONAL CONCEPTS OF BOUNDED VARIATION AND ABSOLUTE CONTINUITY

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Introduction. Let K_0 denote the class of all single-valued, real-valued, continuous functions $f(u)$ in an interval $I : a \leq u \leq b$. Let K_1 be the class of all those functions $f \varepsilon K_0$ that are BV (of bounded variation) on I, and let finally K_2 be the class of all those functions $f \varepsilon K_0$ that are AC (absolutely continuous) in I. It is well known that $K_2 \subset K_1$. In view of the fundamental importance and general utility of the one-dimensional concepts BV and AC in problems involving functions of a single real variable, many efforts were made to develop *two-dimensional concepts of comparable scope.* In this paper we shall be concerned with a line of thought whose origins may be traced back to researches of Geöcze and Banach. Let $f(u)$ be continuous in $I : a \leq u \leq b$. Then $f(u)$ gives rise to a continuous mapping $T : x = f(u), u \varepsilon I$. For each real number x, let $N(x)$ denote the number (possibly infinite) of those points $u \varepsilon I$ that are mapped into the point x by T. In other words, $N(x)$ is the number (possibly infinite) of the points of the set $T^{-1}(x)$. Clearly, since $T(I)$ is a bounded set, $N(x)$ vanishes outside of a certain finite interval $-M \leq x \leq M$. Banach [1] observed that $f(u)$ is BV in I if and only if the multiplicity function $N(x)$ is summable, and that the total variation of $f(u)$ in I is then equal to the integral of $N(x)$ (over the interval $-M \leq u \leq M$, or equivalently over any interval outside of which $N(x)$ vanishes). Furthermore, he gave a characterization of AC (absolutely continuous) functions $f(u)$ in terms of the multiplicity function $N(x)$. Thus in the one-dimensional case the concepts BV and AC admit of a geometrical interpretation in terms of the multiplicity function $N(x)$ associated with the continuous mapping $T : x = f(u), u \varepsilon I$. Banach then proceeded to extend this approach to the two-dimensional case. Let $x(u, v)$, $y(u, v)$ be continuous (real-valued) functions in a Jordan region which we assume, to simplify the language, to coincide with the unit square $Q : 0 \leq u \leq 1, 0 \leq v \leq 1$. These functions give rise to a continuous mapping $T : x = x(u, v), y = y(u, v), (u, v) \varepsilon Q$. For each point (x, y), let $N(x, y)$ be the number, possibly infinite, of the points $(u, v) \varepsilon Q$ that are mapped by T into (x, y). In other words, $N(x, y)$ is the number of points in the set $T^{-1}(x, y)$. In analogy with the one-dimensional case, the mapping T is BV (of bounded variation) in the sense of Banach if and only if the multiplicity function $N(x, y)$ is summable, and the total variation of T is then defined as the double integral of $N(x, y)$. If $N(x, y)$ fails to be summable, then the total variation of T is by definition equal to ∞. Furthermore, in complete analogy with the one-dimensional case, Banach defines a concept AC (absolute continuity) for con-

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