TOPOLOGICAL METHODS IN VALUATION THEORY

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1. **Introduction.** We present in this paper some new theorems and new proofs of known theorems in valuation theory. Our point of view is to seek to exploit topological techniques to a greater extent than has been done hitherto. These techniques are such as to make no distinction between the archimedean and non-archimedean cases, which appear throughout on an equal footing.

The principal result in the paper is Theorem 9: if F is a field complete in a valuation, and A is a complete metric algebraic algebra over F, then A is of bounded degree. In the case where A is itself a field with a valuation extending that of F, this theorem was proved by Ostrowski [15], [16]. The case where F is the field of real numbers and A a commutative Banach algebra was announced by Mazur [14]. The commutative proof which we give is considerably shorter than Ostrowski's; Mazur's proof is not available but it seems likely that it is based on the same sort of category argument as is used here. We are also able to prove the theorem in the non-commutative case, though here the argument is more elaborate and makes heavy use of Jacobson's recently developed structure theory for rings without chain condition ([10], [11] and [12]). As a particular case, we thus answer in the affirmative Mazur's question as to whether his result extends to non-commutative Banach algebras.

Actually this theorem and several others in the paper are proved under a weaker assumption than the existence of a valuation. This assumption, a modification of an idea due to Shafarevich [19], is as follows: if a set is bounded away from 0, then the set of its inverses is bounded (in the sense defined in §2). We say that such a division ring is of type V. In Theorem 3 it is shown that if a field of type V has a neighborhood of 0 consisting of nilpotent elements, then it admits a valuation. Theorems 1 and 2 characterize division rings with valuations, the second characterization being an extension of Shafarevich's theorem to the non-commutative case.

In §3 various preliminary results are obtained; Theorems 5 and 6 may be of independent interest. In §4 we apply the preceding results to give a new proof of the theorem that a locally compact (= bicompact) division ring admits a valuation. In §6 we obtain a general result (Theorem 11) on sequences of polynomials and a series of corollaries.

I am indebted to Professor Artin for permission to incorporate Lemma 1 in this paper. It was presented at a seminar in Princeton during the course of which he gave other unpublished results, one of which led me to Theorem 11. I am also indebted to Dr. Arens for suggesting the vital category argument used in the proof of Theorem 9.

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