SEMI-AUTOMORPHISMS OF RINGS

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In connection with the extension of von Staudt's theorem to projective geometries over a division ring, Ancochea [2] introduced the concept of a *semi-automorphism* of a ring: an additive automorphism $a \to a'$ satisfying

$$(ab)' + (ba)' = a'b' + b'a'. \tag{1}$$

Recently [3] he proved that if A is a simple algebra of characteristic different from 2, then a semi-automorphism of A is either an automorphism or anti-automorphism. This result fails for characteristic 2, since (1) loses most of its strength in that case. In this note we shall obtain an alternative form of Ancochea's result (Theorem 2 below), which is valid for any characteristic; it consists of replacing (1) by

$$(aba)' = a'b'a'. \tag{2}$$

Roughly speaking, (2) is equivalent to (1) for characteristic different from 2 and otherwise stronger (for the precise statements see Lemmas 1 and 2).

As a preliminary, one proves that the mapping induces an automorphism of the center. This we are able to do for any semi-simple ring with unit (Theorem 1), by making use of recent results of Jacobson [4] and Nakayama and Azumaya [5]. While a similar extension of Theorem 2 remains an open question, we are able (Theorem 3) to give a complete result for semi-simple algebras. (Throughout the paper we use the term "algebra" to mean "algebra of finite order".)

Lemma 1. *Let $a \to a'$ be an additive isomorphism of rings A and A' satisfying* (1), *and suppose that $2a' = 0$ in A' implies $a' = 0$. Then* (2) *holds. Also if A has a unit element, it maps into a unit element of A'.*

Proof. From (1) with $a = b$ we obtain $(a^2)' = (a')^2$. Then (1) with $b = a^2$ yields $(a^3)' = (a')^3$, and (2) now follows from the identity

$$2aba = 4(a + b)^3 - (a + 2b)^3 - 3a^3 + 4b^3 - 2(a^2b + ba^2).$$

The final statement is an immediate consequence of (1).

Lemma 2. *Suppose A is a ring with unit element 1 and $a \to a''$ is an additive isomorphism, satisfying* (2), *of A and a ring A'. Then $e = 1''$ is in the center of A', e^2 is a unit element of A', and the mapping $a \to a' = ea''$ satisfies both* (2) *and* (1).

Proof. From (2) with $a = 1$ we have $ece = c$ for every $c \in A'$. In particular, $e^3 = e$, $e^2c = e^3ce = ece = c$, similarly $ce^2 = c$, $ec = e^2ce = ce$. That $a \to a'$

Received March 17, 1947; revision received May 5, 1947.