ALGEBRAIC COHOMOLOGY GROUPS AND LOOPS

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1. **Introduction.** A two-dimensional cocycle of the (multiplicative) group Q with values in the (additive) abelian group G is a function $f(x, y)$ of two variables x, y in Q with values in G which satisfies the identity

$$f(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0. \tag{1.1}$$

There are trivial solutions of this identity; indeed, if $h(x)$ is any function of one variable $x \in Q$ with values in G, the function

$$f_0(x, y) = h(y) - h(xy) + h(x) \tag{1.2}$$

automatically satisfies the equation (1.1). Any cocycle f',

$$f'(x, y) = f(x, y) + f_0(x, y),$$

which is obtained from a cocycle f by adding such a "trivial" cocycle is said to be cohomologous to f.

It is well known that these cocycles can be interpreted as the "factor sets" which arise in central group extensions. Given the cocycle f for which $f(x, 1) = f(1, y) = 0$ one may construct the group E of all pairs (g, x) for $g \in G$ and $x \in Q$ with the multiplication table

$$(g_1, x)(g_2, y) = (g_1 + g_2 + f(x, y), xy).$$

The identity (1.1) insures that this multiplication is associative. The group E contains the subgroup of all pairs $(g, 1)$ with $g \in G$; this subgroup may be identified with G. Furthermore the correspondence $(g, x) \to x$ is a homomorphism ϕ of E onto Q. The pair (E, ϕ) is a central group extension of Q by G, in that ϕ is a homomorphism of E on Q with kernel G in the center of E.

Conversely, one may show that every central group extension of Q by G may be obtained in this fashion from a two-dimensional cocycle, and that cohomologous cocycles give rise to "equivalent" group extensions. Indeed, the group composed of the cohomology classes of two-dimensional cocycles (this is the so-called two-dimensional cohomology group of Q over G) is isomorphic to a suitably defined *group* of central group extensions of Q by G. A more detailed discussion may be found in reference [6] of the bibliography, or in other references cited there.

The topological problem of constructing the cohomology groups of an aspherical space algebraically from the fundamental group of that space leads to the con-

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