THE DIRICHLET DIVISOR PROBLEM

By Richard Bellman

1. Let $d(n)$ denote the number of divisors of n, and consider

$$D(x) = \sum_{n<x} d(n). \tag{1.1}$$

It was proved by Dirichlet that

$$D(x) = x \log x + (2C - 1)x + \Delta(x) \qquad (x \to \infty), \tag{1.2}$$

where $\Delta(x) = O(x^{\frac{1}{2}})$, and C is Euler's constant. Subsequently, it was shown by Voronoi that $\Delta(x) = O(x^{1/3} \log x)$, and the estimate has been continually improved since, although the precise result is still unknown.

In the other direction, it was proved by Hardy [3] that for some constant k,

$$| \Delta(x) | \geq kx^{1/4} \tag{1.3}$$

for an infinity of values of $x \to \infty$, and even that [2]

$$| \Delta(x) | \geq k(x \log x)^{1/4} \log \log x \tag{1.4}$$

for an infinity of $x \to \infty$. A result of this type is called an Ω-result, and (1.4) is written

$$\Delta(x) = \Omega((x \log x)^{1/4} \log \log x). \tag{1.5}$$

If we are interested, not in exceptional values, but in the average value, as in the case of $d(n)$ itself, we have the following result, also due to Hardy [2]

$$\int_1^T | \Delta(x) | \, dx = O(T^{5/4+\epsilon}) \qquad (T \to \infty). \tag{1.6}$$

The result we wish to prove in this paper is a refinement of this result, namely

Theorem 1.

$$\int_1^T \frac{\Delta(x)^2}{x^{3/2}} \, dx \sim c_1 \log T \qquad (T \to \infty), \tag{1.7}$$

where $c_1 \neq 0$ is a constant. Hence $\Delta(x) = \Omega(x^{1/4})$.

The same method as used in the proof of this theorem yields a similar result for the error term in the Gauss circle problem, which concerns itself with the asymptotic order of

$$\sum_{n<x} r(n) = R(x) \tag{1.8}$$

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