TENSOR THEORY OF INVARIANTS FOR THE PROJECTIVE DIFFERENTIAL GEOMETRY OF A RULED SURFACE

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1. **Introduction.** A linear projective space S_3 of points is a four-dimensional manifold of lines. A line family of one, two, or three parameters forms respectively a *ruled surface*, a *line congruence*, or a *line complex*. The projective differential geometry of a ruled surface has been studied extensively by Wilczynski [6] and this same author has introduced a method of studying the line congruence [7]. A complete invariant theory for the differential geometry of the line complex has not yet been constructed. A preliminary survey of the task of setting up a tensor theory for the line complex has indicated the desirability of first considering the one and two parameter line families from a tensor viewpoint. Hence the present paper has resulted from an application of tensor analysis to the simplest case, the ruled surface theory.

The first step in this direction was made by C. M. Cramlet [3] and further contributions were made jointly by A. Barnett and H. Reingold [1] and J. Levine [5]. However, all of these authors interested themselves chiefly in a generalization of the ruled surface equations and hence did not develop in detail the ruled surface equations themselves.

We shall first express the defining differential equations of the ruled surface theory in tensor form, and from these we shall derive by tensor methods the same complete system of invariants and covariants as first found by Wilczynski. Then from a series expansion of the surface in the neighborhood of a generator there will be derived the equations of fundamental covariant loci for whose invariantive analytical display tensor analysis is indispensable.

2. **The basic differential equations.** If $y^\alpha(t)$ and $z^\alpha(t)$, $\alpha = 1, 2, 3, 4$, be regarded as homogeneous coordinates of points P_y and P_z tracing curves C_y and C_z in a linear projective space S_3, the line $L_{yz}(t)$ joining P_y and P_z will generate a ruled surface S as t varies. The condition $D \equiv | y', z', y, z | \neq 0$, where $y' = dy/dt$, states that S is not a developable surface [6; 130] and will henceforth be assumed. Under this assumption it is possible to solve the equations,

$$(2.1)\qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

for the coefficients p and q in terms of $y^\alpha(t)$, $z^\alpha(t)$ and their derivatives. Wilczynski has shown [6; 126] that, conversely, a system of the form (2.1) defines

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