FUNCTIONS SATISFYING CERTAIN PARTIAL DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE AND THEIR REPRESENTATION

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1. **Introduction.** Various investigations have recently been carried out whose main purpose has been to extend the methods of the theory of analytic functions of one complex variable to various other fields. In particular, these efforts have been directed toward the study of functions satisfying partial differential equations.

In this connection, pairs of functions, say (ϕ, ψ), have been considered such that *either*:

a. Both ϕ and ψ satisfy the same linear partial differential equation of elliptic type, say $\mathbf{L}(\phi) = 0$, $\mathbf{L}(\psi) = 0$, and ϕ and ψ are connected by some relations (which are not necessarily linear relations between the derivatives of ϕ and ψ); *or*

b. ϕ, ψ are considered as satisfying a system of linear partial differential equations which are generalizations of the Cauchy-Riemann equations.

In the first case, it is convenient to combine both functions into one complex solution $u = \phi + i\psi$. The main tools for the investigation of these classes of complex solutions are certain integral operators which transform analytic functions of one complex variable into functions belonging to these classes. (In the case where $\mathbf{L}$ is not of elliptic but of hyperbolic type, then instead of analytic functions of one complex variable it is necessary to use differentiable functions of one real variable.) In many instances, these operators preserve various properties of the functions to which they are applied, and by using them it is possible to investigate the functions ϕ and ψ. For details, see [1], [2], [3]. The present paper does not presuppose a knowledge of previous publications.

In the second case, of particular interest seem to be systems of equations connecting ϕ and ψ which can be written in the form

$$\phi_x = l\psi_y, \qquad \phi_y = -l\psi_x, \tag{2.1}$$

$l = l(x, y)$ is a twice differentiable function in the domain under consideration. In this case, the main tools of investigation are those considerations of the theory of functions of one complex variable which are ultimately based on the Cauchy-Riemann equations.

This approach seems to be particularly successful in the case where l *is positive throughout the domain* under consideration. *This assumption will be made throughout the present paper.*

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